

## REMARKS ON A FAMILY OF COMPLEX POLYNOMIALS

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Integral formulae for the coefficients of cyclotomic and polygonal polynomials were recently obtained in [2] and [3]. In this paper, we define and study a family of polynomials depending on an integer sequence  $m_1, \dots, m_n, \dots$ , and on a sequence of complex numbers  $z_1, \dots, z_n, \dots$  of modulus one. We investigate some particular instances such as: extended cyclotomic, extended polygonal-type, and multinomial polynomials, for which we obtain formulae for the coefficients. Some novel related integer sequences are also derived.

### 1. INTRODUCTION

Recall that the  $n$ th cyclotomic polynomial  $\Phi_n$  is defined by

$$(1) \quad \Phi_n(z) = \prod_{\zeta^n=1} (z - \zeta),$$

where  $\zeta$  are the primitive roots of order  $n$  of the unity. Clearly, the degree of  $\Phi_n$  is  $\varphi(n)$ , where  $\varphi$  is the Euler totient function.

Numerous interesting properties of the cyclotomic polynomials and their coefficients have been discovered over more than a hundred years. Polynomials up to  $n < 105$  only have 0, 1 and  $-1$  as coefficients, while  $-2$  first appears as the coefficient of  $z^7$  of  $\Phi_{105}$ . For the early history of such results we refer to the 1936 paper of E. Lehmer [15]. In 1987, J. Suzuki [19] proves that in fact, any integer can be a coefficient of a cyclotomic polynomial of a certain degree. The result was further improved more recently by C.-G. Ji and W.-P. Li [13].

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The explicit computation of the coefficients of cyclotomic polynomials involves elaborated calculations [17, p.258–259]. Recently, integral formulae for the coefficients were established by D. Andrica and O. Bagdasar in [2].

Recall that the  $n$ th *polygonal polynomial* was defined in [3], by the formula

$$(2) \quad P_n(z) = (z - 1)(z^2 - 1) \cdots (z^n - 1).$$

Clearly, the roots of  $P_n$  are the complex coordinates of the vertices, with repetitions, of the regular  $k$ -gons centered in the origin, and having 1 as a vertex,  $k = 1, \dots, n$ , which justifies the name. Explicit formulae and recurrences for the coefficients of the polynomials (2) have been presented in [3], along with links to some partition problems, Mahonian polynomials, and new integer sequences. A general framework based on Cauchy's integral formula and further applications to the study of  $k$ -partitions of multisets have been presented in the paper [4].

In Section 2 of this paper, we introduce and investigate a class of polynomials defined by factorization, whose roots are situated on the unit circle with possible multiplicities. As for the cyclotomic polynomials  $\Phi_n$ , the coefficients in the algebraic expansion generate many interesting problems. As particular cases we obtain the extended cyclotomic, the extended polygonal-type, and the multinomial polynomials.

For these classes of polynomials we establish integral formulae for the coefficients (Section 3), which are useful in the study of the asymptotic behaviour of the coefficients. In the last section we generate certain sequences defined by the number of non-zero coefficients or the maximum coefficient, and the number of irreducible factors over integers of these polynomials. In this process we obtain some novel sequences, not currently indexed in the Online Encyclopedia of Integer Sequences (OEIS), or new meanings to existing entries.

## 2. A FAMILY OF POLYNOMIALS AND SPECIAL CASES

In this section we introduce a class of polynomials defined by an integer sequence  $m_1, \dots, m_n, \dots$  and a sequence  $z_1, \dots, z_n, \dots$  of complex numbers of modulus one, which recovers cyclotomic, polygonal-type, and multinomial polynomials as special cases.

### 2.1 The polynomials $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$

Consider the positive integers  $n, m_1, \dots, m_n$ , and the complex numbers  $z_1, \dots, z_n$  with  $|z_1| = \dots = |z_n| = 1$ . We define the polynomial

$$(3) \quad F_{m_1, \dots, m_n}^{z_1, \dots, z_n}(z) = \prod_{k=1}^n (z^{m_k} - z_k).$$

Clearly, the degree of  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}(z)$  is  $m_1 + \dots + m_n$ . While this polynomial is factorized as a product of factors  $z - \zeta$ , where  $|\zeta| = 1$ , but this form is not practical. The product of absolute values of the roots is one, hence the Mahler measure of  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$  is 1. An interesting and challenging problem is to find reasonable formulae for the coefficients after multiplication.

## 2.2 The extended cyclotomic polynomials

Let  $z_k = \zeta_k$ ,  $k = 1, \dots, \varphi(n)$ , be the  $n$ th primitive roots of unity. Alternatively, these can be indexed as powers of the first primitive root  $\zeta_1 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ , i.e.,  $\zeta_1^j$ , with  $1 \leq j \leq n-1$  and  $\gcd(j, n) = 1$ .

The *extended cyclotomic* polynomial of degree  $m_1 + \dots + m_{\varphi(n)}$  is

$$(4) \quad \Phi_{m_1, \dots, m_{\varphi(n)}}(z) = \prod_{k=1}^{\varphi(n)} (z^{m_k} - \zeta_k) = \prod_{\substack{1 \leq j \leq n-1 \\ \gcd(j, n)=1}} (z^{\tilde{m}_j} - \zeta_1^j),$$

where for  $k = 1, \dots, \varphi(n)$ , we have  $m_k = \tilde{m}_j$ , with  $j$  being the  $k$ th positive integer relatively prime with  $n$ , with  $1 \leq j \leq n-1$ . Generally, this polynomial has complex coefficients.

For  $m_1 = \dots = m_{\varphi(n)} = 1$  we obtain the classical cyclotomic polynomials defined in (1), i.e., we have  $\Phi_{1, \dots, 1} = \Phi_n$ . It is well known that the coefficients  $c_j^{(n)}$ ,  $j = 0, \dots, \varphi(n)$ , of  $\Phi_n$  are integers, while the cyclotomic polynomials are irreducible over  $\mathbb{Z}$  (see for example, [12], Theorem 1, p. 195).

The characterization of extended cyclotomic polynomials with integer coefficients can be done by using Kronecker's Theorem (see, e.g., [9] and [11]). Here we present an alternative proof based on Galois theory.

Consider the extension  $K = \mathbb{Q}(\zeta_n)/\mathbb{Q}$ . This is Galois and the ring of integers of  $K$  is  $O(K) = \mathbb{Z}[\zeta_n]$ ; see, e.g., [14, Theorem 4, Chapter IV]. By the fundamental theorem of Galois theory, an element  $\alpha$  in  $K$  is rational if and only if  $\sigma(\alpha) = \alpha$  for all  $\sigma$  in  $\text{Gal}(K/\mathbb{Q})$  and, as a consequence of the fact that for any number field  $K$  one has  $O(K) \cap \mathbb{Q} = \mathbb{Z}$ , we deduce that an element  $\beta$  in  $O(K)$  is an integer if and only if  $\sigma(\beta) = \beta$  for all  $\sigma$  in  $\text{Gal}(K/\mathbb{Q})$  (see [10, Theorem 7.3.1]).

Denoting for simplicity  $P = \Phi_{m_1, \dots, m_{\varphi(n)}}$ , this polynomial has coefficients in  $O(K)$  and by the previous observations, these coefficients are integers precisely when  $\sigma(P(z)) = P(z)$  for all  $\sigma \in \text{Gal}(K/\mathbb{Q})$ . To finish off, we note that  $\text{Gal}(K/\mathbb{Q})$  acts transitively on the group of roots of unity, and hence, for any  $i, j \in (\mathbb{Z}/n\mathbb{Z})^*$ , there is a  $\sigma$  in  $\text{Gal}(K/\mathbb{Q})$  such that  $\sigma(\zeta_n^i) = \zeta_n^j$ . This shows that  $m_i = m_j$  (the equality of  $P(z)$  and  $\sigma(P(z))$  is one of polynomials in  $K[z]$ , which is a UFD and all the factors appearing are irreducible) and since  $i$  and  $j$  were arbitrary, it shows that  $m_1 = \dots = m_{\varphi(n)} = s$ . In this case we have  $\Phi_{m_1, \dots, m_{\varphi(n)}}(z) = \Phi_n(z^s)$ , where  $s$  is an arbitrary positive integer. Finally, we mention that the polynomials  $\Phi_n(z^s)$  play an important role in the study of cyclotomic partitions.

### 2.3 The extended polygonal-type polynomials

Letting  $z_1 = \cdots = z_n = 1$  in (3), we define the *extended polygonal* polynomial by

$$(5) \quad F_{m_1, \dots, m_n}(z) = \prod_{k=1}^n (z^{m_k} - 1).$$

According to the well-known formula  $x^m - 1 = \prod_{d|m} \Phi_d(x)$ , it follows that

$$F_{m_1, \dots, m_n}(z) = \prod_{k=1}^n \prod_{d|m_k} \Phi_d(z),$$

i.e., the polynomial  $F_{m_1, \dots, m_n}$  has exactly  $\nu(m_1) + \cdots + \nu(m_n)$  irreducible factors over  $\mathbb{Z}$ , where  $\nu(a)$  denotes the number of divisors of  $a$ . Since all the factors are cyclotomic,  $F_{m_1, \dots, m_n}$  is a Kronecker polynomial [9].

The polynomial (5) has integer coefficients, while for  $m_k = k$ ,  $k = 1, \dots, n$ , one recovers the polygonal polynomial (2). The roots of  $z^{m_k} - 1$  are the complex coordinates of the vertices of the regular  $m_k$ -gon centered at the origin, having 1 as a vertex. Clearly, the roots of the polynomial (5) are the complex coordinates of the vertices, with repetitions, of the regular  $m_k$ -gons with  $k = 1, \dots, n$ .

**Theorem 1.** *The number of distinct roots of the polynomial (5) is*

$$(6) \quad \mathcal{R}_{m_1, \dots, m_n} = m_1 + \cdots + m_n + \sum_{j=2}^n (-1)^{j-1} \sum_{1 \leq k_1 < \cdots < k_j \leq n} \gcd(m_{k_1}, \dots, m_{k_j}).$$

*Proof.* For  $n = 2$ , the polynomials  $z^{m_1} - 1$  and  $z^{m_2} - 1$  have  $m_1$  and  $m_2$  distinct roots, respectively. Out of these, a number of  $\gcd(m_1, m_2)$  are common. In general, the proof follows by the inclusion-exclusion principle. Clearly, the result holds even if the numbers  $m_1, \dots, m_n$  are not all distinct.  $\square$

**Remark 2.** *For  $m_k = k$ ,  $k = 1, \dots, n$ , a direct counting leads to*

$$(7) \quad \mathcal{R}_{1, \dots, n} = \sum_{k=1}^n \varphi(k).$$

This represents the sequence A002088 in the Online Encyclopedia of Integer Sequences (OEIS) [16], which has numerous interesting properties.

**Theorem 3.** *Let  $n \geq 2$ , and  $m_1, \dots, m_n$  be integers and consider the set*

$$\mathcal{D}_{m_1, \dots, m_n} = \{d \in \mathbb{N} : d \text{ is a divisor of } m_k, \text{ for some } k = 1, \dots, n\}.$$

*For each  $d \in \mathcal{D}_{m_1, \dots, m_n}$  we denote the number of multiples of  $d$  in  $\mathcal{D}_{m_1, \dots, m_n}$  by  $M_d(\mathcal{D}_{m_1, \dots, m_n})$ . The following identity holds:*

$$(8) \quad \sum_{d \in \mathcal{D}_{m_1, \dots, m_n}} \varphi(d) \cdot M_d(\mathcal{D}_{m_1, \dots, m_n}) = m_1 + \cdots + m_n.$$

*Proof.* We count the roots of the polynomial (5) in two ways, considering multiplicities. A primitive root  $\zeta = \cos \frac{2s\pi}{d} + i \sin \frac{2s\pi}{d}$  of order  $d$  with  $\gcd(s, d) = 1$  is a root of (5) if and only if  $d \in \mathcal{D}_{m_1, \dots, m_n}$ . Indeed, for each  $k = 1, \dots, n$ , the roots of  $z^{m_k} - 1 = 0$  are distinct, while  $\zeta$  is a  $m_k$ th root, whenever  $d$  is a divisor of  $m_k$ .

For each  $d \in \mathcal{D}_{m_1, \dots, m_n}$ , the number of  $d$ th primitive roots of unity is  $\varphi(d)$ , while the multiplicity of each such root in (5) is given by  $M_d(\mathcal{D}_{m_1, \dots, m_n})$ . On the other hand, the polynomial (5) has  $m_1 + \dots + m_n$  roots, counting multiplicities.  $\square$

For  $m_k = k$  for  $k = 1, \dots, n$ , by Theorem 3 we recover the following identity.

**Corollary 4.** *Let  $1 \leq k \leq n$  be an integer. Then*

$$\sum_{k=1}^n \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

*Proof.* Each  $k = 1, \dots, n$  has  $\left\lfloor \frac{n}{k} \right\rfloor$  multiples in the set  $\{1, \dots, n\}$ .  $\square$

Notice that in the limit cases  $m_1 = \dots = m_n$  or  $n = 1$ , formula (8) reduces to the classical summation formula of Gauss  $\sum_{d|m} \varphi(d) = m$ .

For  $z_1 = \dots = z_n = -1$  in (3), we define the *extended anti-polygonal* polynomial by the formula

$$(9) \quad A_{m_1, \dots, m_n}(z) = \prod_{k=1}^n (z^{m_k} + 1).$$

Clearly, this polynomial has integer coefficients. Such polynomials play an important role in the study of partitions. For example, the coefficient of  $z^m$  of the polynomial  $A_{2m_1, \dots, 2m_n}$ , where  $m = m_1 + \dots + m_n$ , is the number of ordered bi-partitions of the set  $\{m_1, \dots, m_n\}$  having equal sums (see, e.g., [1, 6, 7, 8]).

We have that  $F_{m_1, \dots, m_n} \cdot A_{m_1, \dots, m_n} = F_{2m_1, \dots, 2m_n}$ , which is a product of  $\nu(2m_1) + \dots + \nu(2m_n)$  cyclotomic polynomials. The number of irreducible factors over  $\mathbb{Z}$  of the polynomial  $A_{m_1, \dots, m_n}$  is

$$(10) \quad \nu(2m_1) + \dots + \nu(2m_n) - [\nu(m_1) + \dots + \nu(m_n)].$$

Obviously, all these factors are cyclotomic, hence  $A_{m_1, \dots, m_n}$  is a Kronecker polynomial. This is a special case of Kronecker's Theorem [9, Chapter 6, pp.43-56].

## 2.4. The multinomial polynomials

Let  $s, m, m_1, \dots, m_s$  be positive integers with  $m_1 + \dots + m_s = m$ . The classical *multinomial* polynomial can be written using the polygonal polynomials (2) defined as:

$$(11) \quad \binom{m}{m_1, \dots, m_s}_z = \frac{P_m(z)}{P_{m_1}(z) \cdots P_{m_s}(z)} = \sum_{j=0}^M C_j^{m_1, \dots, m_s} z^j.$$

The degree of  $\binom{m}{m_1, \dots, m_s}_z$  is  $M = \frac{1}{2} [m^2 - (m_1^2 + \dots + m_s^2)]$ . It can be shown that this polynomial is recovered as a particular case of (3) (see [4]). For  $s = 2$  and  $m_1 = r$ , one obtains the Gaussian polynomial defined by

$$(12) \quad \binom{m}{r}_z = \frac{P_m(z)}{P_r(z)P_{m-r}(z)} = \begin{cases} \frac{(z^{m-r+1}-1)\dots(z^m-1)}{(z-1)\dots(z^r-1)} & r \leq m \\ 0 & r > m. \end{cases}$$

### 3. INTEGRAL FORMULAE FOR THE COEFFICIENTS

#### 3.1. Coefficients of the polynomial $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$

In what follows we obtain an integral formula for the coefficients of  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$ . The method we use is a special case of the Cauchy integral formula (see [9]), adapted for complex polynomials, where the integration curve is the unit circle (see [4]). Denote  $z_k = \cos \alpha_k + i \sin \alpha_k$ ,  $k = 1, \dots, n$ ,  $\alpha = \alpha_1 + \dots + \alpha_n$ ,  $m = m_1 + \dots + m_n$ , and consider  $z = \cos 2t + i \sin 2t$  for  $t \in [0, \pi]$ .

To get a unified formula for the coefficients of the polynomial (3), it is useful to introduce the function

$$(13) \quad \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) = \prod_{k=1}^n \sin \left( m_k t - \frac{\alpha_k}{2} \right), \quad t \in [0, \pi].$$

For each  $k = 1, \dots, n$ , by computations involving Euler's exponential notation of complex numbers in polar form, we obtain:

$$(14) \quad \begin{aligned} z^{m_k} - z_k &= (\cos 2m_k t - \cos \alpha_k) + i (\sin 2m_k t - \sin \alpha_k) \\ &= 2i \sin \left( m_k t - \frac{\alpha_k}{2} \right) e^{i(m_k t + \frac{\alpha_k}{2})}. \end{aligned}$$

Writing the polynomial  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$  in algebraic form, it follows that

$$(15) \quad \begin{aligned} F_{m_1, \dots, m_n}^{z_1, \dots, z_n}(z) &= \sum_{j=0}^m C_j z^j = \prod_{k=1}^n (z^{m_k} - z_k) = (2i)^n \prod_{k=1}^n \sin \left( m_k t - \frac{\alpha_k}{2} \right) e^{i(m_k t + \frac{\alpha_k}{2})} \\ &= (2i)^n \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) e^{i(mt + \frac{\alpha}{2})}. \end{aligned}$$

Using the multiplication of complex numbers in polar form we deduce

$$\begin{aligned} C_j + \sum_{k \neq j} C_k z^{k-j} &= z^{-j} \prod_{k=1}^n (z^{m_k} - z_k) \\ &= z^{-j} (2i)^n \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) e^{i(mt + \frac{\alpha}{2})} \\ &= (2i)^n \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) e^{i((m-2j)t + \frac{\alpha}{2})}. \end{aligned}$$

Integrating the above relation on the interval  $[0, \pi]$ , we get the following result.

**Theorem 5.** (1) The coefficients of polynomial  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$  are given by

$$(16) \quad C_j = \frac{(2i)^n}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) e^{i((m-2j)t + \frac{\alpha}{2})} dt.$$

(2) If the coefficient  $C_j$  is a real number, then it is given by

$$\begin{cases} \frac{(-1)^{\frac{n}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) \cos((m-2j)t + \frac{\alpha}{2}) dt & \text{if } n \text{ is even} \\ \frac{(-1)^{\frac{n+1}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_n; \alpha_1, \dots, \alpha_n) \sin((m-2j)t + \frac{\alpha}{2}) dt & \text{if } n \text{ is odd.} \end{cases}$$

### 3.2. Coefficients of extended cyclotomic polynomials

As mentioned at the beginning of subsection 2.2, for a coherent notation, the powers will also be indexed by the primitive roots, i.e.,  $m_k$  with  $k \leq n-1$  and  $\gcd(k, n) = 1$ . In this case, for each  $t \in [0, \pi]$  formula (13) becomes

$$\Lambda(t; m_1, \dots, m_{\varphi(n)}; \zeta_1, \dots, \zeta_{\varphi(n)}) = \Lambda(t; m_1, \dots, m_{\varphi(n)}) = \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k, n)=1}} \sin(m_k t - \frac{k\pi}{n}).$$

As shown in [2, Lemma 2.1], in this case we have

$$\alpha = \sum_{\substack{1 \leq k \leq n-1 \\ \gcd(k, n)=1}} \frac{2k\pi}{n} = \varphi(n)\pi.$$

Since  $\varphi(n)$  is even for  $n \geq 3$ , by the relation  $e^{ik\pi} = (-1)^k$ ,  $k \in \mathbb{Z}$ , and Theorem 5 (1) the coefficients of the polynomial  $\Phi_{m_1, \dots, m_{\varphi(n)}}$  are given by

$$(17) \quad \begin{aligned} C_j &= \frac{(-1)^{\frac{\varphi(n)}{2}} 2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_{\varphi(n)}) e^{i((m-2j)t + \frac{\varphi(n)}{2}\pi)} dt \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_{\varphi(n)}) e^{i(m-2j)t} dt. \end{aligned}$$

If the coefficient  $C_j$  is a real number, then by Theorem 5 (2), we have

$$(18) \quad C_j = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_{\varphi(n)}) \cos(m-2j)t dt.$$

As a consequence, since  $\cos(m-2j)t = \cos(m-2(m-j))t$ ,  $t \in [0, 2\pi]$ , from (18) we obtain  $C_j = C_{m-j}$ ,  $j = 0, \dots, m$ , hence we get the following result:

**Corollary 6.** If  $\Phi_{m_1, \dots, m_{\varphi(n)}}$  has real coefficients, then it is reciprocal.

**Remark 7** (Theorem 1 [2]). *If  $m_1 = m_2 = \dots = m_{\varphi(n)} = 1$  we obtain the following integral formula for the coefficients  $c_j^{(n)}$  of the classical cyclotomic polynomial.*

$$(19) \quad c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cos(\varphi(n) - 2j)t \, dt, \quad j = 0, 1, \dots, \varphi(n),$$

where

$$\Lambda_n(t) = \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right).$$

### 3.3. Coefficients of extended polygonal-type polynomials

Notice that for the polynomial  $F_{m_1, \dots, m_n}$  given by formula (5), we have  $z_k = 1$ , hence  $\alpha_k = 0$  for  $k = 1, \dots, n$ , and  $\alpha = \alpha_1 + \dots + \alpha_n = 0$ . We shortly denote by  $\Lambda(t; m_1, \dots, m_n)$ , the function  $\Lambda(t; m_1, \dots, m_n, 0, \dots, 0)$ . We have

$$(20) \quad \Lambda(t; m_1, \dots, m_n) = \prod_{k=1}^n \sin m_k t, \quad t \in [0, \pi].$$

Since the coefficients  $C_j$  of  $F_{m_1, \dots, m_n}$  given by (5) are real, by Theorem 5 (2) we obtain the following result.

**Theorem 8.** *The coefficients of the polynomial  $F_{m_1, \dots, m_n}$  are given by*

$$(21) \quad C_j = \begin{cases} \frac{(-1)^{\frac{n}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_n) \cos(m - 2j)t \, dt & \text{if } n \text{ is even} \\ \frac{(-1)^{\frac{n+1}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; m_1, \dots, m_n) \sin(m - 2j)t \, dt & \text{if } n \text{ is odd.} \end{cases}$$

Setting  $m_k = k$  for  $k = 1, \dots, n$ , we obtain the polygonal polynomials  $P_n$  given in (2), investigated in our paper [3]. In this case we have

$$(22) \quad \Lambda(t; 1, \dots, n) = \prod_{k=1}^n \sin kt, \quad t \in [0, \pi],$$

and  $m = 1 + \dots + n = \frac{n(n+1)}{2}$ . Even if this polynomial seems simple, there are very few things known about its coefficients  $c_j^{(n)}$ ,  $j = 0, \dots, \frac{n(n+1)}{2}$ . Using formula (21), the coefficients are given by the following explicit formula:

$$c_j^{(n)} = \begin{cases} \frac{(-1)^{\frac{n}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; 1, \dots, n) \cos\left(\frac{n(n+1)}{2} - 2j\right)t \, dt & \text{if } n \text{ is even} \\ \frac{(-1)^{\frac{n+1}{2}} 2^n}{\pi} \int_0^\pi \Lambda(t; 1, \dots, n) \sin\left(\frac{n(n+1)}{2} - 2j\right)t \, dt & \text{if } n \text{ is odd.} \end{cases}$$

For the polynomial  $A_{m_1, \dots, m_n}$  defined by (9) one has  $z_k = -1$ , hence  $\alpha_k = \pi$  for  $k = 1, \dots, n$  and  $\alpha = \alpha_1 + \dots + \alpha_n = n\pi$ . We denote  $\Lambda(t; m_1, \dots, m_n, \pi, \dots, \pi)$



by  $\tilde{\Lambda}(t; m_1, \dots, m_n)$ , and obtain

$$(23) \quad \tilde{\Lambda}(t; m_1, \dots, m_n) = \prod_{k=1}^n \cos m_k t, \quad t \in [0, \pi].$$

Using this notation, the coefficients of the polynomial  $A_{m_1, \dots, m_n}$  are given below.

**Theorem 9.** *The coefficients of the polynomial  $A_{m_1, \dots, m_n}$  are given by*

$$(24) \quad C_j = \frac{2^n}{\pi} \int_0^\pi \tilde{\Lambda}(t; m_1, \dots, m_n) \cos(m-2j)t \, dt, \quad j = 0, \dots, m.$$

*Proof.* The antipolygonal polynomial (9) can be written as

$$(25) \quad \begin{aligned} A_{m_1, \dots, m_n}(z) &= \sum_{j=0}^m C_j z^j = \prod_{k=1}^n (z^{m_k} + 1) = (2i)^n \prod_{k=1}^n \sin\left(m_k t - \frac{\pi}{2}\right) e^{i(m_k t + \frac{\pi}{2})} \\ &= (2i)^n (-1)^n (i^n) \left( \prod_{k=1}^n \cos m_k t \right) e^{imt} = 2^n \tilde{\Lambda}(t; m_1, \dots, m_n) e^{imt}. \end{aligned}$$

We deduce that for  $j = 0, \dots, m$ , we have

$$\begin{aligned} C_j + \sum_{k \neq j} C_k z^{k-j} &= z^{-j} \prod_{k=1}^n (z^{m_k} + 1) = z^{-j} 2^n \tilde{\Lambda}(t; m_1, \dots, m_n) e^{imt} \\ &= 2^n \tilde{\Lambda}(t; m_1, \dots, m_n) e^{i(m-2j)t}. \end{aligned}$$

Integrating the above relation on the interval  $[0, \pi]$ , and since the coefficients  $C_j$ ,  $j = 0, \dots, m$ , are real numbers, we get the main result. It also follows that

$$\int_0^\pi \tilde{\Lambda}(t; m_1, \dots, m_n) \sin(m-2j)t \, dt = 0, \quad j = 0, \dots, m.$$

□

**Remark 10.** *An interesting particular case is obtained for  $m_k = k$ ,  $k = 1, \dots, n$ . If  $S(n)$  is the number of ordered bipartitions of  $\{1, 2, \dots, n\}$  into sets having equal sums, then this is the middle coefficient of the polynomial  $A_{1, 2, \dots, n}$ , that is the coefficient of  $z^{\frac{n(n+1)}{2}}$  when  $n \equiv 0$  or  $n \equiv 3 \pmod{4}$ . In the paper [7], D. Andrica and I. Tomescu conjectured the asymptotic formula*

$$S(n) \sim \sqrt{\frac{6}{\pi}} \cdot \frac{2^n}{n\sqrt{n}},$$

where  $f(n) \sim g(n)$  means that  $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$ . This formula has been proven by B. D. Sullivan [18], using the corresponding integral formula from Theorem 9 and complicated analytic techniques.

### 3.4. Coefficients of the multinomial polynomials

Consider the function defined by

$$(26) \quad \Lambda_{m_1, \dots, m_s}^m(t) = \frac{\prod_{k=1}^m \sin kt}{\prod_{j=1}^s (\prod_{k=1}^{m_j} \sin kt)}.$$

By Theorem 5 (2) we get an integral formula for the coefficients of  $\binom{m}{m_1, \dots, m_s}_z$ .

**Proposition 11** (Theorem 7 [5]). *The coefficients  $C_j^{m_1, \dots, m_s}$  of  $\binom{m}{m_1, \dots, m_s}_z$  are given by*

$$(27) \quad C_j^{m_1, \dots, m_s} = \frac{1}{\pi} \int_0^\pi \Lambda_{m_1, \dots, m_s}^m(t) \cdot \cos(M - 2j)t \, dt, \quad j = 0, \dots, M.$$

The middle coefficient of  $\binom{m}{m_1, \dots, m_s}_z$  depends on the parity of  $M$ .

**Proposition 12** (Proposition 9 [5]). *1. If  $M = 2k$ , then*

$$(28) \quad C_k^{m_1, \dots, m_s} = \frac{1}{\pi} \int_0^\pi \Lambda_{m_1, \dots, m_s}^m(t) \, dt.$$

*2. If  $M = 2k + 1$ , then*

$$(29) \quad C_k^{m_1, \dots, m_s} = C_{k+1}^{m_1, \dots, m_s} = \frac{1}{\pi} \int_0^\pi \Lambda_{m_1, \dots, m_s}^m(t) \cdot \cos t \, dt.$$

When  $n = 2$ , one obtains the coefficients of the Gaussian polynomial.

**Proposition 13** (Theorem 1 [5]). *Consider the integers  $r, m$  such that  $m \geq 2$  and  $0 \leq r \leq m$ . The coefficients of the Gaussian polynomial  $\binom{m}{r}_z$  are given by*

$$(30) \quad C_j^{(m,r)} = \frac{1}{\pi} \int_0^\pi \Lambda_{r, m-r}^m(t) \cdot \cos[r(m-r) - 2j]t \, dt, \quad j = 0, \dots, r(m-r).$$

In [5], the authors have also discussed further results concerning multinomial, Gaussian, and Catalan polynomials and their coefficients.

## 4. CONNECTIONS TO SOME INTEGER SEQUENCES

For a fixed  $n \geq 1$ , the polynomial  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$ , given by (5), has the degree  $m_1 + \dots + m_n$ , hence it has  $m_1 + \dots + m_n + 1$  coefficients. In this section we discuss some integer sequences related to  $F_{m_1, \dots, m_n}^{z_1, \dots, z_n}$ , for particular choices of the sequences  $m_1, \dots, m_n, \dots$  and  $z_1, \dots, z_n, \dots$ .

The coefficients of  $\Phi_{m_1, \dots, m_{\varphi(n)}}$ , are generally complex. For example, for  $n = 3$ ,  $\zeta = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ ,  $m_1 = 2$  and  $m_2 = 3$ , we obtain the polynomial

$$\Phi_{2,3}(z) = (z^2 - \zeta)(z^3 - \zeta^2) = z^5 - \zeta z^3 - \zeta^2 z^2 + 1.$$

The extended polygonal polynomial  $F_{m_1, \dots, m_n}$  has integer coefficients. For example, when  $n = 4$  and  $m_1 = 1$ ,  $m_2 = 2$ ,  $m_3 = 3$  and  $m_4 = 4$  one has

$$F_{1,2,3,4}(z) = (z-1)(z^2-1)(z^3-1)(z^4-1) = z^{10} - z^9 - z^8 + 2z^5 - z^2 - z + 1,$$

which has 7 non-zero coefficients of which the highest is equal to 2.

In this section we explore some integer sequences related to the non-zero coefficients of the polynomials  $\Phi_{m_1, \dots, m_{\varphi(n)}}$  and  $F_{m_1, \dots, m_n}$ , the maximum coefficient for the sequence  $F_{m_1, \dots, m_n}$ , as well as to the number of irreducible factors over integers for the polynomial  $A_{m_1, \dots, m_n}$ . The numerical calculations producing the sequences presented in the Tables 1-5 in this section, have been computed in Matlab® and Wolfram Alpha. In this process we recover some new integer sequences, not currently indexed in OEIS [16].

#### 4.1. The number of non-zero coefficients of $\Phi_{m_1, \dots, m_{\varphi(n)}}$

Here, we discuss the number of non-zero coefficients for some particular choices. Considering  $m_k = 1$  for  $k = 1, \dots, \varphi(n)$ , we obtain the classical cyclotomic polynomial  $\Phi_{1, \dots, 1} = \Phi_n$ . The coefficients of  $\Phi_n$  are integers and they have been studied in detail by numerous authors.

For example, setting  $n = 5$  we have  $\varphi(5) = 4$  and  $\zeta = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ , hence

$$\Phi_{1,1,1,1}(z) = \Phi_4(z) = (z-\zeta)(z-\zeta^2)(z-\zeta^3)(z-\zeta^4) = z^4 + z^3 + z^2 + z + 1.$$

Also, setting  $m_1 = 1$  and  $m_k = 2$  for  $k \geq 2$ , for  $n = 5$  we obtain

$$\begin{aligned} \Phi_{1,2,2,2}(z) &= (z-\zeta)(z^2-\zeta^2)(z^2-\zeta^3)(z^2-\zeta^4) = z^7 - \zeta z^6 - (\zeta^2 + \zeta^3 + \zeta^4)z^5 \\ &\quad + (1 + \zeta^3 + \zeta^4)z^4 + (1 + \zeta + \zeta^2)z^3 - (\zeta + \zeta^2 + \zeta^3)z^2 - \zeta^4 z + 1. \end{aligned}$$

When  $n = 6$ ,  $\varphi(6) = 2$ ,  $\zeta = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6}$  for  $m_1 = 1$ ,  $m_2 = 2$  we get

$$\Phi_{1,2}(z) = (z-\zeta)(z^2-\zeta^5) = z^3 - \zeta z^2 - \zeta^5 z + 1.$$

Also, for  $m_1 = 1$ ,  $m_2 = 5$ , where  $\Phi_{1,5}(z) = (z-\zeta)(z^5-\zeta^5) = z^6 - \zeta z^5 - \zeta^5 z + 1$ .

Some interesting integer sequences generated by the extended cyclotomic polynomials are defined by the number of non-zero coefficients.

A few examples are presented in Table 1 below. The first row recovers the sequence of non-zero coefficients of cyclotomic polynomials, which are indexed as A051664 in OEIS. The second row corresponds to the sequence A140434, which represents the number of new visible points created at each step in an  $n \times n$  grid ( $n \geq 2$ ). One may notice that the results in the last two rows are similar whenever  $n$  is a prime, while both sequences are not currently indexed in the Encyclopedia. Here, we have denoted by  $p_n$ , the  $n$ th prime number.

Polynomial	Sequence of non-zero coefficients	OEIS
$\Phi_{1,\dots,1}$	2, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, 7, 7, 2, 17, ...	A051664
$\Phi_{1,2,\dots,2}$	2, 2, 4, 4, 8, 4, 12, 8, 12, 8, 20, 8, 24, 12, 16, 16, ...	A140434
$\Phi_{1,2,3,\dots,3}$	2, 4, 4, 4, 10, 4, 16, 10, 16, 10, 28, 10, 34, 16, 22, ...	NEW
$\Phi_{1,2,\dots,\varphi(n)}$	2, 2, 4, 4, 11, 4, 22, 11, 22, 11, 56, 11, 79, 22, 37, ...	NEW
$\Phi_{j_1,j_2,\dots,j_{\varphi(n)}}$	2, 2, 4, 4, 11, 4, 22, 15, 28, 15, 56, 15, 79, 39, 61, ...	NEW

Table 1: The number of non-zero coefficients of the polynomial  $\Phi_{m_1,\dots,m_{\varphi(n)}}$ ,  $n \geq 1$ . The numbers  $j_1 < \dots < j_k < \dots < j_{\varphi(n)}$  are relatively prime with  $n$ .

#### 4.2. The number of non-zero coefficients of $F_{m_1,\dots,m_n}$

The sequence giving the number of non-zero coefficients of  $F_{m_1,\dots,m_n}$  ( $n \geq 1$ ) recovers some known integer sequences, as well as a novel sequence when  $m_n = p_n$  is the  $n$ th prime ( $n \geq 1$ ), as seen in Table 2. The first and third lines are identical, since  $F_{km_1,\dots,km_n}(z) = F_{m_1,\dots,m_n}(z^k)$ , for all  $k \in \mathbb{N}$ .

Polynomial	Sequence of non-zero coefficients	OEIS
$F_{1,\dots,n}$	2, 4, 6, 7, 12, 14, 18, 25, 32, 36, 42, 53, 68, 64, 84, ...	A086781
$F_{1,\dots,2n-1}$	2, 4, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, ...	A082562
$F_{2,\dots,2n}$	2, 4, 6, 7, 12, 14, 18, 25, 32, 36, 42, 53, 68, 64, 84, ...	A086781
$F_{1,\dots,n^2}$	2, 4, 8, 16, 24, 40, 68, 103, 162, 236, 344, 453, 612, ...	A225549
$F_{p_1,\dots,p_n}$	2, 4, 6, 8, 14, 20, 24, 32, 46, 66, 92, 138, 162, 204, ...	NEW

Table 2: The number of non-zero coefficients of the polynomial  $F_{m_1,\dots,m_n}$ ,  $n \geq 1$ .

#### 4.3. The maximum coefficient of $F_{m_1,\dots,m_n}$

The maximum coefficient of  $F_{m_1,\dots,m_n}$  ( $n \geq 1$ ) recovers some known integer sequences when  $m_n = n$  or  $m_n = 2n$ , while it also generates some novel sequences in the other cases, as seen in Table 3.

Polynomial	Sequence of maximum coefficients of $F_{m_1,\dots,m_n}$	OEIS
$F_{1,\dots,n}$	1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 4, 3,3, 4, 6, 5, 6, 7, 8, ...	A086376
$F_{1,\dots,2n-1}$	1, 1, 1, 2, 2, 3, 5, 8, 13, 22, 38, 68, 118, 211, 380, ...	NEW
$F_{2,\dots,2n}$	1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 4, 3, 3, 4, 6, 5, 6, 7, 8, ...	A086376
$F_{1,\dots,n^2}$	1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 4, 6, 7, 8, 11, 14, 12, 12, ...	NEW
$F_{p_1,\dots,p_n}$	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 4, 6, 8, 12, 17, 30, ...	NEW

Table 3: The maximum coefficient of the polynomial  $F_{m_1,\dots,m_n}$ ,  $n \geq 1$ .

#### 4.4. The number of irreducible factors over integers of $A_{m_1, \dots, m_n}$

The number of irreducible factors of  $A_{m_1, \dots, m_n}$  given by (10), simplifies to  $\sum_{k=1}^n \frac{\nu(m_k)}{\alpha_k + 1}$ , where  $m_k = 2^{\alpha_k} m'_k$ , with  $m'_k$  odd,  $k = 1, \dots, n$ . Some sequences arise naturally for particular choices of  $m_k$ ,  $k = 1, \dots, n$ .

Polynomial	Number of irreducible factors	OEIS
$z^n + 1$	1, 1, 2, 1, 2, 2, 2, 1, 3, 2, 2, 2, 2, 2, 4, ...	A001227
$z^{3n} + 1$	2, 2, 3, 2, 4, 3, 4, 2, 4, 4, 4, 3, 4, 4, 6, ...	NEW
$z^{9n} + 1$	3, 3, 4, 3, 6, 4, 6, 3, 5, 6, 6, 4, 6, 6, 8, ...	NEW

Table 4: Number of irreducible factors over the integers of the polynomial  $z^{m_n} + 1$ , for  $m_n = n$ ,  $m_n = 3n$  and  $m_n = 9n$ ,  $n \geq 1$ .

The interpretations known for the OEIS sequence A001227 in the first row of Table 4 include: the number of odd divisors of  $n$ , the number of ways to write  $n$  as difference of two triangular numbers, the number of partitions of  $n$  into consecutive positive integers (including the trivial one of length 1), or the number of factors in the factorization of the  $n$ th Chebyshev polynomial of the first kind.

Polynomial	Number of irreducible factors of $A_{m_1, \dots, m_n}$	OEIS
$A_{1, \dots, n}$	1, 2, 4, 5, 7, 9, 11, 12, 15, 17, 19, 21, 23, 25, 29, ...	A060831
$A_{3, \dots, 3n}$	2, 4, 7, 9, 13, 16, 20, 22, 26, 30, 34, 37, 41, 45, 51, ...	NEW
$A_{9, \dots, 9n}$	3, 6, 10, 13, 19, 23, 29, 32, 37, 43, 49, 53, 59, 65, 73, ...	NEW

Table 5: The number of irreducible factors over the integers of the polynomial  $A_{m_1, \dots, m_n}$ , for  $m_n = n$ ,  $m_n = 3n$  and  $m_n = 9n$ ,  $n \geq 1$ .

Some interpretations known for the OEIS sequence A060831 in the first row of Table 5 are the following: the number of odd divisors present in  $\{1, \dots, n\}$ , the number of sums less than or equal to  $n$  of sequences of consecutive positive integers (including sequences of length 1), or the total number of partitions of all positive integers less or equal to  $n$  into an odd number of equal parts. The asymptotic formula for the OEIS sequence A060831 was conjectured in 2019 as  $a(n) \sim n(\log(2n) + 2\gamma - 1)/2$ , where  $\gamma$  is the Euler-Mascheroni constant.

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