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# ON GENERALISED MULTI-INDEX NON-LINEAR RECURSION IDENTITIES FOR TERMS OF THE HORADAM SEQUENCE

Peter J. Larcombe and Eric J. Fennessey

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**Abstract**. We state and prove a non-linear recurrence identity for terms of the so called Horadam sequence, and then offer its generalisation which is available from the same methodology. We illustrate how the overarching idea may be used to sequentially produce extended versions, each in turn with an extra level of non-linearity and term index complexity. These identities can all be captured in matrix determinant form.

## 1 Introduction

Consider the second order linear recursion

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a, w_1 = b, \tag{1.1}$$

which, characterised by the four parameters $a, b, p, q$, defines the long established Horadam sequence—written $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$—dating back to the 1960s [1]. Roots of the characteristic polynomial $\lambda^2 - p\lambda + q$ for (1.1) give rise to separate degenerate ($p^2 = 4q$) and non-degenerate ($p^2 \neq 4q$) case closed forms for $w_n$ whose constructions are standard undergraduate exercises. For $p^2 \neq 4q$ ($p, q \neq 0$), there are two distinct characteristic roots

$$\alpha(p, q) = (p + \sqrt{p^2 - 4q})/2, \qquad \beta(p, q) = (p - \sqrt{p^2 - 4q})/2, \tag{1.2}$$

and, for $n \geq 0$, a closed form

$$w_n(a, b; p, q) = w_n(\alpha(p, q), \beta(p, q), a, b) = \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta} \tag{1.3}$$

accordingly. For $p^2 = 4q$, on the other hand, the characteristic roots co-incide as simply

$$\alpha(p) = \beta(p) = p/2, \tag{1.4}$$

and, for $n \geq 0$, it is found that

$$w_n(a, b; p, p^2/4) = w_n(\alpha(p), a, b) = bn\alpha^{n-1} - a(n - 1)\alpha^n; \tag{1.5}$$

the relations

$$\alpha + \beta = p, \qquad \alpha\beta = q, \tag{1.6}$$

cover both root types.

In this paper we derive a non-linear recurrence identity for Horadam sequence terms, and generalise it to a form that we believe is as yet unseen in the literature. The matrix method deployed—which contrasts with the technique applied in a recent paper published by the authors [1] on a similar topic—hinges on a particular matrix (called an 'insertion' matrix) that introduces a pair of ('free') additional indices, attached to sequence terms, into the final identity version. We explain how multiple insertion matrices lead to further variants of increased complexity, and provide some examples where we see that each identity may be expressed neatly through matrix determinants.

## 2 A Result and Proof

We begin by deriving a non-linear recurrence identity for terms of the Horadam sequence.

**Identity.** For $r, s \geq 0$,

$$(b^2 - a[pb - qa])w_{r+s} = w_{r+1}(bw_s - aw_{s+1}) + w_r(bw_{s+1} - [pb - qa]w_s).$$

*Proof.* Let

$$\mathbf{H}(p,q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{I.1}$$

from which the recursion (1.1) readily delivers the matrix power relations

$$\begin{pmatrix} w_{r+1} \\ w_r \end{pmatrix} = \mathbf{H}^r(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_{r+2} \\ w_{r+1} \end{pmatrix} = \mathbf{H}^r(p,q) \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}, \tag{I.2}$$

that hold for $r \geq 0$. We introduce further a matrix

$$\mathbf{W} = \mathbf{W}(w_0, w_1, w_2) = \begin{pmatrix} w_1 & w_2 \\ w_0 & w_1 \end{pmatrix}, \tag{I.3}$$

involving the first three Horadam sequence terms, whose determinant $W = W(w_0, w_1, w_2) = |\mathbf{W}| = (w_1)^2 - w_0 w_2$ is assumed non-zero. Denoting the 2-square identity matrix as $\mathbf{I}_2$ we proceed as follows, writing (and using (I.3),(I.2) where needed)

$$\begin{aligned}
&\begin{pmatrix} w_{r+s+1} \\ w_{r+s} \end{pmatrix} \\
&= \mathbf{H}^{r+s}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \mathbf{H}^r(p,q)\mathbf{I}_2\mathbf{H}^s(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \mathbf{H}^r(p,q)\mathbf{W}(w_0,w_1,w_2)\mathbf{W}^{-1}(w_0,w_1,w_2)\mathbf{H}^s(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \frac{1}{W(w_0,w_1,w_2)}\mathbf{H}^r(p,q) \begin{pmatrix} w_1 & w_2 \\ w_0 & w_1 \end{pmatrix} \begin{pmatrix} w_1 & -w_2 \\ -w_0 & w_1 \end{pmatrix} \mathbf{H}^s(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \frac{1}{W(w_0,w_1,w_2)} \begin{pmatrix} w_{r+1} & w_{r+2} \\ w_r & w_{r+1} \end{pmatrix} \begin{pmatrix} w_1 & -w_2 \\ -w_0 & w_1 \end{pmatrix} \begin{pmatrix} w_{s+1} \\ w_s \end{pmatrix} \\
&\vdots \\
&= \frac{1}{W(w_0,w_1,w_2)} \begin{pmatrix} w_{r+1}(w_1 w_{s+1} - w_2 w_s) + w_{r+2}(w_1 w_s - w_0 w_{s+1}) \\ w_r(w_1 w_{s+1} - w_2 w_s) + w_{r+1}(w_1 w_s - w_0 w_{s+1}) \end{pmatrix},
\end{aligned} \tag{I.4}$$

after a little algebra. With $w_2 = pw_1 - qw_0 = pb - qa$ by (1.1), the identity is immediate. □

We note that the condition $0 \neq W(w_0, w_1, w_2) = (w_1)^2 - w_0 w_2 = b^2 - a(pb - qa)$ is equivalent to $b/a \neq \alpha(p,q), \beta(p,q)$, for we see that $(w_1)^2 - w_0 w_2 = (w_1)^2 - w_0(pw_1 - qw_0) = (w_1)^2 - pw_0 w_1 + q(w_0)^2 = (w_1)^2 - (\alpha + \beta)w_0 w_1 + \alpha\beta(w_0)^2$ (by (1.6)) $= (w_1 - \alpha w_0)(w_1 - \beta w_0)$.

## 3 A Generalisation

The formulation detailed allows itself to be generalised somewhat by simply deploying a matrix

$$\mathbf{W}_i = \mathbf{W}_i(w_i, w_{i+1}, w_{i+2}) = \begin{pmatrix} w_{i+1} & w_{i+2} \\ w_i & w_{i+1} \end{pmatrix} \tag{3.1}$$

where $i \geq 0$ is any integer. We leave it to the interested reader to re-work the above proof and arrive at a more general identity of which the previous is merely the $i = 0$ instance (in which case the 'insertion' matrix $\mathbf{W}_0$ (3.1) co-incides with $\mathbf{W}$ (I.3)). It is found that (for every integer $i \geq 0$ such that $w_{i+1}(a, b; p, q)/w_i(a, b; p, q)$ takes a value which is neither $\alpha(p, q)$ nor $\beta(p, q)$), for $r, s \geq 0$ also, $[(w_{i+1})^2 - w_i w_{i+2}]w_{r+s} = w_{r+i+1}(w_{i+1}w_s - w_i w_{s+1}) + w_{r+i}(w_{i+1}w_{s+1} - w_{i+2}w_s)$. Clearly, however, we can go further if we use the *most general* insertion matrix possible, namely,

$$\mathbf{W}_{i,j} = \mathbf{W}_{i,j}(w_i, w_j, w_{i+1}, w_{j+1}) = \begin{pmatrix} w_{i+1} & w_{j+1} \\ w_i & w_j \end{pmatrix}, \tag{3.2}$$

whereupon we have a finalised result which is seemingly new to the literature:

**Identity I (Generalised: Four Index).** For every integer $i, j \geq 0$ such that $w_{i+1}(a, b; p, q)/w_i(a, b; p, q) \neq w_{j+1}(a, b; p, q)/w_j(a, b; p, q)$ (this removes multiplicity in the columns of $\mathbf{W}_{i,j}$),

$$(w_{i+1}w_j - w_i w_{j+1})w_{r+s} = w_{r+i}(w_j w_{s+1} - w_{j+1}w_s) - w_{r+j}(w_i w_{s+1} - w_{i+1}w_s)$$

for $r, s \geq 0$ also.

This deeper result has been verified using each of the closed forms (1.3) and (1.5) by algebraic computation. In so far as we can regard the Horadam elements $w_i, w_{i+1}$ and $w_j, w_{j+1}$ as having free index variables this last result describes a set (or class) of such identities, although it is perhaps more natural to think of it as a single four index recursion (in $i, j, r, s$) which is symmetric in $i, j$. A feature is the absence of any of the basic characterising Horadam sequence parameters $a, b, p, q$ explicitly which, of course, is a consequence of the methodology and carries over to its natural re-application in generating higher level results.

## 4 Extension to Higher Order Results

### 4.1 Re-Application of Methodology

The formulation presented lends itself to sequential re-application, each adding a layer of depth through the appearance of additional indices in the resulting identity. Writing $T$ for transposition then, expressing the vector $(w_{v_1+v_2+\cdots+v_n+1}, w_{v_1+v_2+\cdots+v_n})^T$ as $\mathbf{H}^{v_1}(p, q)\mathbf{I}_2\mathbf{H}^{v_2}(p, q)\mathbf{I}_2 \cdot \cdots \cdot \mathbf{I}_2\mathbf{H}^{v_n}(p, q)(w_1, w_0)^T$, each of the $n - 1$ identity matrices can be replaced by a product $\mathbf{W}^{(p)}(\mathbf{W}^{(p)})^{-1}$ (based on a $p$th insertion matrix $\mathbf{W}^{(p)}$) where, for $p = 1, \ldots, n - 1$, $\mathbf{W}^{(p)} = [w_{i_{2p-1}+1}, w_{i_{2p}+1}|w_{i_{2p-1}}, w_{i_{2p}}]$ and introduces a pair of indices $i_{2p-1}, i_{2p}$. This then yields an identity containing the $n + 2(n - 1) = 3n - 2$ indices $v_1, v_2, \ldots, v_n, i_1, i_2, \ldots, i_{2(n-1)}$ (with $n - 1$ accompanying conditions ensuring the existence of $(\mathbf{W}^{(1)})^{-1}, (\mathbf{W}^{(2)})^{-1}, \ldots, (\mathbf{W}^{(n-1)})^{-1}$); we observe that Identity I above is but the $n = 2$ case (the smallest value of $n$) for which $v_1, v_2, i_1, i_2$ are $r, s, i, j$, having utilised a single insertion matrix $[w_{i_1+1}, w_{i_2+1}|w_{i_1}, w_{i_2}] = [w_{i+1}, w_{j+1}|w_i, w_j] = \mathbf{W}^{(1)} = \mathbf{W}_{i,j}$ of (3.2). As an example, the next identity in the series is a seven index one (valid for the $3(3) - 2 = 7$ indices in the case $n = 3$—where $v_1, v_2, v_3, i_1, i_2, i_3, i_4$ are, in order, $r, s, t, i, j, k, l$— using two insertion matrices) which is certainly non-trivial.

**Identity II (Seven Index).** For every integer $i, j, k, l \geq 0$ (for $r, s, t \geq 0$ too) such that $w_{i+1}/w_i \neq w_{j+1}/w_j$ and $w_{k+1}/w_k \neq w_{l+1}/w_l$,

$$\begin{aligned}
(w_{i+1}w_j - w_i w_{j+1})(w_{k+1}w_l - w_k w_{l+1})w_{r+s+t} = \\
w_{r+k}\{w_l[w_{s+i+1}(w_j w_{t+1} - w_{j+1}w_t) - w_{s+j+1}(w_i w_{t+1} - w_{i+1}w_t)] \\
- w_{l+1}[w_{s+i}(w_j w_{t+1} - w_{j+1}w_t) - w_{s+j}(w_i w_{t+1} - w_{i+1}w_t)]\} - \\
w_{r+l}\{w_k[w_{s+i+1}(w_j w_{t+1} - w_{j+1}w_t) - w_{s+j+1}(w_i w_{t+1} - w_{i+1}w_t)] \\
- w_{k+1}[w_{s+i}(w_j w_{t+1} - w_{j+1}w_t) - w_{s+j}(w_i w_{t+1} - w_{i+1}w_t)]\}.
\end{aligned}$$

Again, this has been checked algebraically using computer software. As an aside, it has the virtue of offering an interesting special case

$$(w_{k+1}w_l - w_k w_{l+1})w_{r+u} = w_{u+1}(w_{r+k}w_l - w_{r+l}w_k) - w_u(w_{r+k}w_{l+1} - w_{r+l}w_{k+1}), \tag{4.1}$$

having set $i = t$ and replaced $s + t$ with $u$.

### 4.2 Determinant Forms

It is interesting to see that Identities I and II may be re-cast as, resp.,

$$\begin{vmatrix} w_{i+1} & w_{j+1} \\ w_i & w_j \end{vmatrix} w_{r+s} = \begin{vmatrix} w_{r+i} & w_{r+j} & 0 \\ w_i & w_j & w_s \\ w_{i+1} & w_{j+1} & w_{s+1} \end{vmatrix} \tag{4.2}$$

and

$$\begin{vmatrix} w_{i+1} & w_{j+1} \\ w_i & w_j \end{vmatrix} \begin{vmatrix} w_{k+1} & w_{l+1} \\ w_k & w_l \end{vmatrix} w_{r+s+t} = \begin{vmatrix} w_{r+k} & w_{r+l} & 0 & 0 & 0 \\ w_k & w_l & w_{s+i} & w_{s+j} & 0 \\ w_{k+1} & w_{l+1} & w_{s+i+1} & w_{s+j+1} & 0 \\ 0 & 0 & w_i & w_j & w_t \\ 0 & 0 & w_{i+1} & w_{j+1} & w_{t+1} \end{vmatrix}, \tag{4.3}$$

while the next (ten index) one in the set—based on three insertion matrices—is captured as

$$\begin{vmatrix} w_{i+1} & w_{j+1} \\ w_i & w_j \end{vmatrix} \begin{vmatrix} w_{k+1} & w_{l+1} \\ w_k & w_l \end{vmatrix} \begin{vmatrix} w_{m+1} & w_{n+1} \\ w_m & w_n \end{vmatrix} w_{r+s+t+u}$$
$$= \begin{vmatrix} w_{r+m} & w_{r+n} & 0 & 0 & 0 & 0 & 0 \\ w_m & w_n & w_{s+k} & w_{s+l} & 0 & 0 & 0 \\ w_{m+1} & w_{n+1} & w_{s+k+1} & w_{s+l+1} & 0 & 0 & 0 \\ 0 & 0 & w_k & w_l & w_{t+i} & w_{t+j} & 0 \\ 0 & 0 & w_{k+1} & w_{l+1} & w_{t+i+1} & w_{t+j+1} & 0 \\ 0 & 0 & 0 & 0 & w_i & w_j & w_u \\ 0 & 0 & 0 & 0 & w_{i+1} & w_{j+1} & w_{u+1} \end{vmatrix}, \tag{4.4}$$

where all index variables are $\geq 0$. Beyond this, we have similarly (with thirteen indices),

$$\begin{vmatrix} w_{i+1} & w_{j+1} \\ w_i & w_j \end{vmatrix} \begin{vmatrix} w_{k+1} & w_{l+1} \\ w_k & w_l \end{vmatrix} \begin{vmatrix} w_{m+1} & w_{n+1} \\ w_m & w_n \end{vmatrix} \begin{vmatrix} w_{o+1} & w_{p+1} \\ w_o & w_p \end{vmatrix} w_{r+s+t+u+v} =$$
$$\begin{vmatrix} w_{r+o} & w_{r+p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_o & w_p & w_{s+m} & w_{s+n} & 0 & 0 & 0 & 0 & 0 \\ w_{o+1} & w_{p+1} & w_{s+m+1} & w_{s+n+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_m & w_n & w_{t+k} & w_{t+l} & 0 & 0 & 0 \\ 0 & 0 & w_{m+1} & w_{n+1} & w_{t+k+1} & w_{t+l+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_k & w_l & w_{u+i} & w_{u+j} & 0 \\ 0 & 0 & 0 & 0 & w_{k+1} & w_{l+1} & w_{u+i+1} & w_{u+j+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_i & w_j & w_v \\ 0 & 0 & 0 & 0 & 0 & 0 & w_{i+1} & w_{j+1} & w_{v+1} \end{vmatrix} \tag{4.5}$$

(where the index $p$ here is used locally in this context only).[1] The phenomenon is both pleasing and perhaps not so surprising, describing in a succinct way the inherent 'fractal' nature of our identities' r.h.s. algebra as complexity levels increase (and where blocks of algebraic terms become nested as part of more lengthy expressions).

[1] The pairwise symmetry of these identities in $i$ and $j$, $k$ and $l$, $m$ and $n$, and so on, is reflected in the interchangeability of neighbouring columns of the matrices comprising them (for the determinant of a matrix changes sign under a column swap).

## 5 Summary

In this paper we have described a means to generate new, and highly non-linear, multi-index recurrence identities for Horadam terms that retain the four defining sequence parameters $a, b, p, q$; the results are, therefore, completely generalised ones. The standardised approach we have taken manifests itself in the structure evident in their determinant forms which are both concise and visually striking (and, while not given explicitly, are generalisable to a single statement[2] that accounts for those $n = 2, 3, 4, 5$ instances included here and others of higher specific order).

## References

[1] P. J. Larcombe and E. J. Fennessey, A new non-linear recurrence identity class for Horadam sequence terms, *Palest. J. Math.* **7**, 406–409.

### Author information

Peter J. Larcombe, Department of Computing and Mathematics, College of Engineering and Technology, University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
E-mail: `p.j.larcombe@derby.ac.uk`

Eric J. Fennessey, BAE Systems Surface Ships Ltd., HM Naval Base, Portsmouth PO1 3NJ, U.K.
E-mail: `eric.fennessey@baesystems.com`

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[2]In which $\left\{\prod_{p=1}^{n-1}\begin{vmatrix} w_{i_{2p-1}+1} & w_{i_{2p}+1} \\ w_{i_{2p-1}} & w_{i_{2p}} \end{vmatrix}\right\} w_{v_1+\cdots+v_n}$ is evaluated as the determinant of a dimension $2n-1$ matrix.