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ON TWO DERIVATIVE SEQUENCES FROM SCALED GEOMETRIC MEAN SEQUENCE TERMS

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Abstract The so called geometric mean sequence recurrence, with additional scaling variable, produces a sequence for which the general term has a known closed form. Two types of derivative sequence—comprising products of such sequence terms—are examined. In particular, the general term closed forms formulated are shown to depend strongly on a mix of three existing sequences, from which sequence growth rates are deduced and other results given.

1 Introduction

1.1 Background

Consider the (scaled) geometric mean sequence $\{g_n(a,b;c)\}_{n=0}^{\infty} = \{g_n(a,b;c)\}_0^{\infty}$ defined, given $g_0 = a$, $g_1 = b$, through the recurrence

$$g_{n+1} = c\sqrt{g_n g_{n-1}}, \qquad n \geq 1, \tag{1.1}$$

where $a, b, c \in \mathbb{Z}^+$, the variable c playing the role of a scaling constant. It delivers the sequence

$$\{g_n(a,b;c)\}_0^{\infty} = \{a, b, (abc^2)^{\frac{1}{2}}, (ab^3c^6)^{\frac{1}{4}}, (a^3b^5c^{18})^{\frac{1}{8}}, (a^5b^{11}c^{46})^{\frac{1}{16}}, (a^{11}b^{21}c^{114})^{\frac{1}{32}}, \ldots\}. \tag{1.2}$$

We begin by introducing here, at the outset, those sequences that play an essential role in our results and which exhibit an element of commonality within the structure to their general term closed forms; this is no co-incidence, as we shall see. Firstly, the Jacobsthal sequence

$$\{J_n\}_0^{\infty} = \{J_0, J_1, J_2, J_3, J_4, \ldots\} = \{0, 1, 1, 3, 5, \ldots\} \tag{1.3}$$

is Sequence No. A001045 on the OEIS [5] and has a well known $(n+1)$th term closed form

$$J_n = [2^n - (-1)^n]/3, \qquad n \geq 0, \tag{1.4}$$

while Sequence No. A045883 is

$$\{S_n\}_0^{\infty} = \{S_0, S_1, S_2, S_3, S_4, \ldots\} = \{0, 1, 3, 9, 23, \ldots\}, \tag{1.5}$$

for which

$$S_n = [(3n+1)2^n - (-1)^n]/9, \qquad n \geq 0. \tag{1.6}$$

Finally, we write

$$\{T_n\}_0^{\infty} = \{T_0, T_1, T_2, T_3, T_4, \ldots\} = \{1, 5, 19, 61, 179, \ldots\} \tag{1.7}$$

to denote Sequence No. A102841, where

$$T_n = [(9n^2 + 33n + 26)2^n + (-1)^n]/27, \qquad n \geq 0; \tag{1.8}$$

note, within similar formats, the increasing order (resp., 0,1,2) of the polynomial multiplier of the element 2^n in each of (1.4),(1.6) and (1.8).

It is known that

$$g_n(a,b;c) = (a^{J_{n-1}} b^{J_n} c^{2S_{n-1}})^{2^{-(n-1)}}, \qquad n \geq 1, \tag{1.9}$$

from a formulation in [3] using standard difference equations theory applied to (1.1) (see Theorem A.1 in the Appendix thereof); the $c = 1$ version is given earlier in that paper and obtained using a different method. Equation (1.1) is the instance $p = q = \frac{1}{2}$ of the more general recursion $z_{n+1} = c(z_n)^p(z_{n-1})^q$ ($n \geq 1$; $z_0 = a$, $z_1 = b$) analysed by Larcombe and Fennessey in [1] and [2]; the feature of these works is the breakdown of closed forms for the sequence general term according to naturally arising cases governed by conditions on the power parameters p, q. For more on the background to this particular topic—which has its origins in a small empiric observation by M.W. Bunder from the mid 1970s—the reader is referred to those references in [1] or [3].

1.2 This Paper

We use (1.9) as the basis to derive closed forms for product expressions involving $g_n(a,b;c)$, namely (for $n \geq 1$),

$$P_n(a,b;c) = \prod_{i=1}^{n} [g_i(a,b;c)]^{2^{i-1}} \tag{1.10}$$

and

$$Q_n(a,b;c) = \prod_{i=1}^{n} g_i(a,b;c). \tag{1.11}$$

Growth rates of the sequences $\{P_n(a,b;c)\}_1^\infty$ and $\{Q_n(a,b;c)\}_1^\infty$ follow in consequence, and some new inter-connections between the sequences $\{J_n\}_0^\infty, \{S_n\}_0^\infty$ and $\{T_n\}_0^\infty$ are found.

2 Results and Analysis

2.1 The Product $P_n(a,b;c)$

Consider $P_n(a,b;c)$ as defined in (1.10). We prove the following:

Theorem 2.1. *For* $n \geq 1$,

$$P_n(a,b;c) = a^{(J_{n+1}-1)/2} b^{(J_{n+2}-1)/2} c^{2(S_n+S_{n+1})-S_{n+2}+1}.$$

Noting that, by (1.9),

$$P_n(a,b;c) = \prod_{i=1}^{n} a^{J_{i-1}} b^{J_i} c^{2S_{i-1}}, \tag{2.1}$$

we write $P_n(a,b;c) = a^{A(n)} b^{B(n)} c^{C(n)}$ and deduce $A(n), B(n), C(n)$ in line with Theorem 2.1, where clearly $A(n) = \sum_{i=1}^{n} J_{i-1}$, $B(n) = \sum_{i=1}^{n} J_i$ and $C(n) = 2\sum_{i=1}^{n} S_{i-1}$ from (2.1) by inspection. It is, perhaps, instructive to see first how the initial elements of the sequence

$$\{P_n(a,b;c)\}_1^\infty = \{b, ab^2c^2, a^2b^5c^8, a^5b^{10}c^{26}, a^{10}b^{21}c^{72}, a^{21}b^{42}c^{186}, \ldots\} \tag{2.2}$$

are generated through (2.1) as

$$\begin{aligned}
P_1(a,b;c) &= a^{J_0}b^{J_1}c^{2S_0} = a^0b^1c^0 = b,\\
P_2(a,b;c) &= P_1(a,b;c)\cdot a^{J_1}b^{J_2}c^{2S_1} = b\cdot a^1b^1c^2 = ab^2c^2,\\
P_3(a,b;c) &= P_2(a,b;c)\cdot a^{J_2}b^{J_3}c^{2S_2} = ab^2c^2\cdot a^1b^3c^6 = a^2b^5c^8,\\
P_4(a,b;c) &= P_3(a,b;c)\cdot a^{J_3}b^{J_4}c^{2S_3} = a^2b^5c^8\cdot a^3b^5c^{18} = a^5b^{10}c^{26},\\
P_5(a,b;c) &= P_4(a,b;c)\cdot a^{J_4}b^{J_5}c^{2S_4} = a^5b^{10}c^{26}\cdot a^5b^{11}c^{46} = a^{10}b^{21}c^{72},\\
P_6(a,b;c) &= P_5(a,b;c)\cdot a^{J_5}b^{J_6}c^{2S_5} = a^{10}b^{21}c^{72}\cdot a^{11}b^{21}c^{114} = a^{21}b^{42}c^{186},
\end{aligned} \tag{2.3}$$

and so on, allowing Theorem 2.1 to be readily checked in advance for a few low values of n by hand—as an example, the exponents in $P_5(a, b; c) = a^{10}b^{21}c^{72}$ are given by $A(5) = (J_6 - 1)/2 = (21 - 1)/2 = 10$, $B(5) = (J_7 - 1)/2 = (43 - 1)/2 = 21$, and $C(5) = 2(S_5 + S_6) - S_7 + 1 = 2(57 + 135) - 313 + 1 = 72$; other automated computations have, of course, been run to check the result exhaustively.

Proof. We begin our proof of Theorem 2.1 with different formulations (for interest) of a simple result which yields, in the form required, both $A(n)$ and $B(n)$. The exponent function $C(n)$ needs a separate treatment, not surprisingly.

Lemma 2.2. *For* $n \geq 0$,

$$\sum_{i=0}^{n} J_i = \frac{1}{2}(J_{n+2} - 1).$$

Method I (Via Series Summations).

Proof. We have, using (1.4), $3\sum_{i=0}^{n} J_i = \sum_{i=0}^{n} 2^i - \sum_{i=0}^{n}(-1)^i = 2^{n+1} - 1 - \frac{1}{2}[1 - (-1)^{n+1}]$ (summing both geometric series) $= 2^{n+1} - \frac{3}{2} - \frac{1}{2}(-1)^{n+2} = \frac{1}{2}[2^{n+2} - (-1)^{n+2}] - \frac{3}{2} = \frac{1}{2}\cdot 3J_{n+2} - \frac{3}{2}$ (by (1.4) again) $= \frac{3}{2}(J_{n+2} - 1)$, and the result follows. □

Method II (Via Telescoping).

Proof. Since it is known that

$$J_{n+1} = J_n + 2J_{n-1}, \qquad n \geq 1, \tag{L.1}$$

we may write $2\sum_{i=0}^{n} J_i = \sum_{i=0}^{n}(J_{i+2} - J_{i+1}) = (J_2 - J_1) + (J_3 - J_2) + (J_4 - J_3) + \cdots + (J_{n+1} - J_n) + (J_{n+2} - J_{n+1}) = J_{n+2} - J_1 = J_{n+2} - 1$, and we have our result. □

Method III (Via Induction).

Proof. Lemma 2.2 holds trivially at $n = 0$ (we see that $\sum_{i=0}^{0} J_i = J_0 = 0 = \frac{1}{2}(1 - 1) = \frac{1}{2}(J_2 - 1)$). Assume, therefore, that it is true for some $n = k \geq 0$, for which $\sum_{i=0}^{k} J_i = \frac{1}{2}(J_{k+2} - 1)$, and consider $\sum_{i=0}^{k+1} J_i = \sum_{i=0}^{k} J_i + J_{k+1} = \frac{1}{2}(J_{k+2} - 1) + J_{k+1}$ (using the inductive hypothesis) $= \frac{1}{2}J_{k+2} + J_{k+1} - \frac{1}{2} = \frac{1}{2}J_{k+3} - \frac{1}{2}$ (by (L.1)) $= \frac{1}{2}(J_{k+3} - 1)$, and the inductive step is upheld. □

From Lemma 2.2, therefore,

$$B(n) = \sum_{i=1}^{n} J_i = \sum_{i=0}^{n} J_i = \frac{1}{2}(J_{n+2} - 1) \tag{P.1}$$

and

$$A(n) = \sum_{i=1}^{n} J_{i-1} = \sum_{i=0}^{n-1} J_i = \frac{1}{2}(J_{n+1} - 1) \tag{P.2}$$

are immediate.

Formulating $C(n)$ requires a little more work, once more adopting a telescoping approach for convenience. We use a known (order three) linear recursion

$$S_n = 3S_{n-1} - 4S_{n-3}, \qquad n \geq 3, \tag{P.3}$$

for the terms of $\{S_n\}_0^\infty$ to write $4\sum_{i=0}^{n} S_i = \sum_{i=0}^{n}(3S_{i+2} - S_{i+3}) = (3S_2 - S_3) + (3S_3 - S_4) + (3S_4 - S_5) + \cdots + (3S_n - S_{n+1}) + (3S_{n+1} - S_{n+2}) + (3S_{n+2} - S_{n+3}) = 3S_2 + 2(S_3 + S_4 + S_5 + \cdots + S_{n+1} + S_{n+2}) - S_{n+3} = 9 + 2\sum_{i=3}^{n+2} S_i - S_{n+3}$. Noting that $\sum_{i=3}^{n+2} S_i = \sum_{i=0}^{n} S_i + S_{n+1} + S_{n+2} - (S_0 + S_1 + S_2) = \sum_{i=0}^{n} S_i + S_{n+1} + S_{n+2} - 4$, then by substitution

$$4\sum_{i=0}^{n} S_i = 9 + 2\left(\sum_{i=0}^{n} S_i + S_{n+1} + S_{n+2} - 4\right) - S_{n+3}, \tag{P.4}$$

so that, re-arranging,

$$2\sum_{i=0}^{n} S_i = 2(S_{n+1} + S_{n+2}) - S_{n+3} + 1, \tag{P.5}$$

and in turn

$$C(n) = 2\sum_{i=1}^{n} S_{i-1} = 2\sum_{i=0}^{n-1} S_i = 2(S_n + S_{n+1}) - S_{n+2} + 1; \tag{P.6}$$

this completes the proof of Theorem 2.1. □

An alternative proof of (P.5) (and so of $C(n)$) is given for the keen reader.

Alternative Proof of (P.5) (Via Induction).

Proof. Equation (P.5) holds trivially at $n = 0$ (the l.h.s. is $2S_0 = 0 = 2(1+3) - 9 + 1 = 2(S_1 + S_2) - S_3 + 1 =$ r.h.s.). Assume, therefore, that it is true for some $n = k \geq 0$, for which $2\sum_{i=0}^{k} S_i = 2(S_{k+1} + S_{k+2}) - S_{k+3} + 1$, and consider $2\sum_{i=0}^{k+1} S_i = 2(\sum_{i=0}^{k} S_i + S_{k+1}) = 2\sum_{i=0}^{k} S_i + 2S_{k+1} = 2(S_{k+1} + S_{k+2}) - S_{k+3} + 1 + 2S_{k+1}$ (by assumption) $= 4S_{k+1} + 2S_{k+2} - S_{k+3} + 1$. Applying (P.3) then $= (3S_{k+3} - S_{k+4}) + 2S_{k+2} - S_{k+3} + 1 = 2(S_{k+2} + S_{k+3}) - S_{k+4} + 1$, as required. □

Aside from the obvious relation $B(n) = A(n+1)$ (observable in (2.2)) we note, in passing, that the exponent functions $A(n), B(n), C(n)$ are readily connected as

$$\begin{aligned} B(n) - A(n) &= J_n, \\ 3C(n) - 2A(n) &= 4[(n-2)2^{n-1} + 1], \\ 3C(n) - 2B(n) &= 4[(n-2)2^{n-1} + 1] - 2J_n, \end{aligned} \tag{2.4}$$

being left as a straightforward reader exercise to check. The first relation is trivial to see by (L.1). The derivation of the second relation is of interest since it follows from the closed form

$$C(n) = 2[(n-2)2^n + (J_{n+1} + 3)/2]/3 \tag{2.5}$$

that arrives for $C(n)$ on consideration of $\sum_{i=0}^{n} S_i = \frac{1}{9}\sum_{i=0}^{n}[(3i+1)2^i - (-1)^i]$, whose series are evaluated and the relation $\frac{1}{2}(J_{n+1} + 3) = A(n) + 2$ (see (P.2)) then used. Equation (2.5) is quite different from (P.6), clearly. By way of illustration, $C(5)$ is correctly given as $C(5) = \frac{2}{3}[3 \cdot 2^5 + \frac{1}{2}(J_6 + 3)] = \frac{2}{3}[3 \cdot 32 + \frac{1}{2}(21 + 3)] = \frac{2}{3}[96 + 12] = 72$. The third relation follows in consequence, instantly, from the other two.

Reconciliation of (P.6) and (2.5) offers a hitherto unseen identity connecting terms of the sequence $\{S_n\}_0^\infty$ with those of the Jacobsthal sequence thus:

$$3[2(S_n + S_{n+1}) - S_{n+2}] - J_{n+1} = (n-2)2^{n+1}, \qquad n \geq 1; \tag{2.6}$$

this is most easily verified (via some simple algebra) not by appeal to the closed forms (1.4) and (1.6), but rather by employing the relation

$$3S_n = n2^n + J_n \tag{2.7}$$

(which holds for $n \geq 0$ and follows therefrom) in combination with (L.1).

We finish Section 2.1 by remarking that the unbounded growth (in all three variables a, b, c) of the sequence $\{P_n(a, b; c)\}_1^\infty$ suggests the sequence growth rate is unbounded also, which is easily deduced as a corollary. Noting, from (2.1), that $P_{n+1}(a, b; c)/P_n(a, b; c) = a^{J_n} b^{J_{n+1}} c^{2S_n}$, define a function $f(n) = [ab^2c^{2(n+1/3)}]^{2^n/3}$. Then we find that $[P_{n+1}(a, b; c)/P_n(a, b; c)]/f(n) = a^{-\frac{1}{3}(-1)^n} b^{\frac{1}{3}(-1)^n} c^{-\frac{2}{9}(-1)^n}$ (whose exponents are bounded) using (1.4),(1.6), and takes one of two values (depending on the parity of n) v_1, v_2, say, where $v_1 \leq v_2$ w.l.o.g. (the ratio oscillates between $[b/(ac^{2/3})]^{\pm 1/3}$, but these coincide when $b/(ac^{2/3})$ is unity—examples being when $a = 3, b = 12, c = 8$, or $a = 2, b = 18, c = 27$). In other words, $v_1 f(n) \leq P_{n+1}(a, b; c)/P_n(a, b; c) \leq v_2 f(n)$, and since $f(n)$ is unbounded for large n we can write the following:

Corollary 2.3. *For large n the ratio $P_{n+1}(a, b; c)/P_n(a, b; c)$ is unbounded if $1 < abc$, and the sequence $\{P_n(a, b; c)\}_1^\infty$ has a growth rate $\lim_{n\to\infty}\{P_{n+1}(a, b; c)/\ P_n(a, b; c)\}$ that is unbounded.*

2.2 The Product $Q_n(a, b; c)$

Consider $Q_n(a, b; c)$ as defined in (1.11). We prove the following:

Theorem 2.4. *For* $n \geq 1$,

$$Q_n(a, b; c) = (a^{S_{n-1}} b^{S_n} c^{2T_{n-2}})^{2^{-(n-1)}}.$$

We write $Q_n(a, b; c) = a^{A^*(n)} b^{B^*(n)} c^{C^*(n)}$, deducing $A^*(n), B^*(n), C^*(n)$ in line with Theorem 2.4 where, from (1.9), $A^*(n) = \sum_{i=1}^{n} J_{i-1}/2^{i-1}$, $B^*(n) = \sum_{i=1}^{n} J_i/2^{i-1}$ and $C^*(n) = \sum_{i=1}^{n} S_{i-1}/2^{i-2}$.

Proof. Consider $3A^*(n) = \sum_{i=1}^{n}[1 - (-\frac{1}{2})^{i-1}]$ (using the closed form (1.4) for J_n) $= n - \sum_{i=1}^{n}(-\frac{1}{2})^{i-1} = n - \frac{2}{3}[1 - (-\frac{1}{2})^n]$, so that

$$A^*(n) = \frac{1}{3}\left\{n - \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]\right\} = \frac{1}{3}\left(n - \frac{J_n}{2^{n-1}}\right), \tag{P.7}$$

again by (1.4). It is known that, for $n \geq 1$, neighbouring Jacobsthal numbers are related according to $J_n + J_{n-1} = 2^{n-1}$, allowing us to further write

$$\begin{aligned} A^*(n) &= \frac{1}{3}\left[n - \frac{1}{2^{n-1}}(2^{n-1} - J_{n-1})\right] \\ &= \frac{1}{2^{n-1}} \cdot \frac{1}{3}[(n-1)2^{n-1} + J_{n-1}] \\ &= S_{n-1}/2^{n-1}, \end{aligned} \tag{P.8}$$

by (2.7).

In a similar fashion (again deploying (1.4) twice), we see that $3B^*(n)/2 = \sum_{i=1}^{n}[1 - (-\frac{1}{2})^i] = n + \frac{1}{3}[1 - (-\frac{1}{2})^n]$, from which it follows that $B^*(n) = \frac{2}{3}(n + J_n/2^n)$, and in turn

$$\begin{aligned} B^*(n) &= \frac{1}{2^{n-1}} \cdot \frac{1}{3}(n2^n + J_n) \\ &= S_n/2^{n-1}, \end{aligned} \tag{P.9}$$

once more by (2.7).

To obtain $C^*(n)$ we write, from (1.6), $9C^*(n)/2 = \sum_{i=1}^{n}[(3i-2) - (-\frac{1}{2})^{i-1}] = 3 \cdot \frac{1}{2}n(n+1) - 2n - \sum_{i=1}^{n}(-\frac{1}{2})^{i-1} = \frac{1}{2}n(3n-1) - \frac{2}{3}[1 - (-\frac{1}{2})^n]$, or

$$C^*(n) = \frac{1}{9}\left[n(3n-1) - \frac{J_n}{2^{n-2}}\right] \tag{P.10}$$

by (1.4); a final simplified form

$$C^*(n) = T_{n-2}/2^{n-2} \tag{P.11}$$

follows, and completes the proof of Theorem 2.4, after further manipulation with reference to (1.8). □

Clearly $B^*(n) = 2A^*(n+1)$ and we note, too, that the exponent functions $A^*(n), B^*(n)$ and $C^*(n)$ are connected as

$$\begin{aligned} B^*(n) - 2A^*(n) &= J_n/2^{n-1}, \\ 3C^*(n) - 2A^*(n) &= n(n-1), \\ 3C^*(n) - B^*(n) &= n(n-1) - J_n/2^{n-1}, \end{aligned} \tag{2.8}$$

in addition to which the relation

$$T_{n+1} - 2T_n = S_{n+2}, \qquad n \geq 0, \tag{2.9}$$

holds and is easily verified using (1.6),(1.8); equation (2.7) now also gives

$$T_{n+1} - 2T_n = [(n+2)2^{n+2} + J_{n+2}]/3, \qquad n \geq 0. \tag{2.10}$$

Other identities are available, for equating (P.10) and (P.11) yields

$$J_n = n(3n-1)2^{n-2} - 9T_{n-2}, \qquad n \geq 2, \tag{2.11}$$

and by (2.7) it reads instead

$$S_n = n(n+1)2^{n-2} - 3T_{n-2}, \qquad n \geq 2. \tag{2.12}$$

We end with a counterpart to Corollary 2.3.

Corollary 2.5. *For large n, $Q_{n+1}(a,b;c)/Q_n(a,b;c) = g_{n+1}(a,b;c) = (a^{J_n}b^{J_{n+1}}c^{2S_n})^{2^{-n}} \sim [ab^2c^{2(n+1/3)}]^{\frac{1}{3}}$, and the sequence $\{Q_n(a,b;c)\}_1^\infty$ has an unbounded growth rate caused by the monotonically increasing exponent of the scaling variable c; if $c=1$ the growth rate becomes finite, taking value $(ab^2)^{\frac{1}{3}}$.*

Remark 2.6. Note that the result $\lim_{n\to\infty}\{g_n(a,b;1)\} = (ab^2)^{\frac{1}{3}}$ recovers the 2005 observation made by Maynard [4, Corollary 1, p. 274] as a special case in his work on a version of the more general power product recurrence alluded to in Section 1.1 here (that is, for $c=1$ and arbitrary recursion powers). To be precise, Maynard considers a sequence $\{z_n(a,b,\beta,\alpha)\}_1^\infty$ generated by the recurrence $z_n = (z_{n-1})^\beta (z_{n-2})^\alpha$ ($n \geq 3$; $z_1 = a$, $z_2 = b$), with Part (iv) of Theorem 1 in [4] stating that (see p. 271) $\lim_{n\to\infty}\{z_n(a,b,1-\alpha,\alpha)\} = (a^\alpha b)^{1/(1+\alpha)}$. The instance $\alpha = 1/2$ yields $\lim_{n\to\infty}\{z_n(a,b,1/2,1/2)\} = \lim_{n\to\infty}\{g_n(a,b;1)\} = (a^{1/2}b)^{1/(3/2)} = (ab^2)^{1/3}$.

2.3 Identities Using Generating Functions

In this short subsection we find further (easily checked) identities, and recover (2.9), using the (ordinary) generating functions

$$\begin{aligned} G_J(x) &= x/[(1+x)(1-2x)] = \sum_{n=0}^{\infty} J_n x^n, \\ G_S(x) &= x/[(1+x)(1-2x)^2] = \sum_{n=0}^{\infty} S_n x^n, \\ G_T(x) &= 1/[(1+x)(1-2x)^3] = \sum_{n=0}^{\infty} T_n x^n, \end{aligned} \tag{2.13}$$

for the respective sequences $\{J_n\}_0^\infty, \{S_n\}_0^\infty$ and $\{T_n\}_0^\infty$.

First, we write

$$G_J(x) = (1-2x)G_S(x), \tag{2.14}$$

giving $\sum_{n=0}^\infty J_n x^n = (1-2x)\sum_{n=0}^\infty S_n x^n = \sum_{n=0}^\infty S_n x^n - 2\sum_{n=0}^\infty S_n x^{n+1}$, or (since $J_0 = S_0 = 0$) $\sum_{n=1}^\infty J_n x^n = \sum_{n=1}^\infty S_n x^n - 2\sum_{n=1}^\infty S_{n-1}x^n = \sum_{n=1}^\infty (S_n - 2S_{n-1})x^n$, from which the identity

$$J_n = S_n - 2S_{n-1}, \qquad n \geq 1, \tag{2.15}$$

follows trivially; together with (2.7) this gives

$$S_n + S_{n-1} = n2^{n-1}, \qquad n \geq 1. \tag{2.16}$$

The relation

$$G_S(x) = x(1-2x)G_T(x) \tag{2.17}$$

merely reproduces (2.9), but the route to it is of interest. With $\sum_{n=0}^\infty S_n x^n = G_S(x) = (x - 2x^2)G_T(x) = \sum_{n=0}^\infty T_n x^{n+1} - 2\sum_{n=0}^\infty T_n x^{n+2}$, we manipulate this as follows: $\sum_{n=1}^\infty S_n x^n = \sum_{n=1}^\infty T_{n-1}x^n - 2\sum_{n=2}^\infty T_{n-2}x^n \Rightarrow S_1 x + \sum_{n=2}^\infty S_n x^n = T_0 x + \sum_{n=2}^\infty T_{n-1}x^n - 2\sum_{n=2}^\infty T_{n-2}x^n$

$= T_0x + \sum_{n=2}^{\infty}(T_{n-1} - 2T_{n-2})x^n$, from which we infer (along with, correctly, $S_1 = T_0$) that, for $n \geq 2$, $S_n = T_{n-1} - 2T_{n-2}$, or (2.9); this identity, when back-substituted into (2.15), yields

$$J_n = T_{n-1} - 4(T_{n-2} - T_{n-3}), \qquad n \geq 3, \tag{2.18}$$

immediately.

Furnished with (2.18) scope to develop other results is considerable, with an obvious one

$$n(3n-1)2^{n-2} = T_{n-1} + 5T_{n-2} + 4T_{n-3}, \qquad n \geq 3, \tag{2.19}$$

using (2.11). Equations (2.9) and (2.12) offer

$$n(n+1)2^{n-2} = T_{n-1} + T_{n-2}, \qquad n \geq 2. \tag{2.20}$$

We leave it to the enthusiastic reader to derive identities of a yet more exotic nature. As a point of interest, we offer in the Appendix an independent re-formulation of $B(n)$ and $A^*(n)$ based on the use of generating functions in (2.13).

2.4 A Periodicity Condition

There exists an instance $b = b(a)$ that produces, with $c = 1$, period 2 cyclicity in both sequences $\{P_n(a, b(a); 1)\}_1^{\infty}$ and $\{P_{n+1}(a, b(a); 1)/P_n(a, b(a); 1)\}_1^{\infty}$. Choosing $b(a) = 1/\sqrt{a}$ consider, for $1 \neq a > 0$, the ratio

$$\frac{P_{n+1}(a, 1/\sqrt{a}; 1)}{P_n(a, 1/\sqrt{a}; 1)} = a^{J_n - J_{n+1}/2} = a^{(-1)^{n+1}/2}, \qquad n \geq 1, \tag{2.21}$$

by (2.1) and (1.4), with

$$\{P_{n+1}(a, 1/\sqrt{a}; 1)/P_n(a, 1/\sqrt{a}; 1)\}_1^{\infty} = \{\sqrt{a}, 1/\sqrt{a}, \ldots\}. \tag{2.22}$$

This also follows from the fact that $P_n(a, 1/\sqrt{a}; 1) = a^{\alpha(n)}$, where $\alpha(n) = \frac{1}{2}(J_{n+1} - 1) - \frac{1}{4}(J_{n+2} - 1)$ (using Theorem 2.1) $= \frac{1}{2}(J_{n+1} - \frac{1}{2}J_{n+2}) - \frac{1}{4} = \frac{1}{4}[(-1)^n - 1]$ (by (1.4) again), so that

$$\{P_n(a, 1/\sqrt{a}; 1)\}_1^{\infty} = \{1/\sqrt{a}, 1, \ldots\} \tag{2.23}$$

is itself of period 2.

No such periodicity is induced in the corresponding sequences

$$\{Q_n(a, 1/\sqrt{a}; 1)\}_1^{\infty} = \{a^{-J_n/2^n}\}_1^{\infty} \tag{2.24}$$

or

$$\{Q_{n+1}(a, 1/\sqrt{a}; 1)/Q_n(a, 1/\sqrt{a}; 1)\}_1^{\infty} = \{a^{(-1/2)^{n+1}}\}_1^{\infty}. \tag{2.25}$$

Remark 2.7. As an aside, the equations $ab^2 = \pm 1$ yield two real and two complex solutions $b(a) = \pm 1/\sqrt{a}, \pm i/\sqrt{a}$ which each offer period 2 sequences thus (containing, of course, (2.22) and (2.23)):

$$\begin{aligned}
\{P_n(a, \pm 1/\sqrt{a}; 1)\}_1^{\infty} &= \{\pm 1/\sqrt{a}, 1, \ldots\}, \\
\{P_n(a, \pm i/\sqrt{a}; 1)\}_1^{\infty} &= \{\pm i/\sqrt{a}, -1, \ldots\}, \\
\{P_{n+1}(a, \pm 1/\sqrt{a}; 1)/P_n(a, \pm 1/\sqrt{a}; 1)\}_1^{\infty} &= \{\pm\sqrt{a}, \pm 1/\sqrt{a}, \ldots\}, \\
\{P_{n+1}(a, \pm i/\sqrt{a}; 1)/P_n(a, \pm i/\sqrt{a}; 1)\}_1^{\infty} &= \{\pm\sqrt{a}i, \mp i/\sqrt{a}, \ldots\}.
\end{aligned} \tag{2.26}$$

We do not pursue cyclicity any further in the paper.

3 Summary

Terms taken from the (scaled) geometric mean sequence are used to construct two types of product sequence, with a closed form found for the general term of each and sequence growth rates examined. An interesting aspect of the results is, within the closed forms formulated, the appearance of three existing O.E.I.S. sequences, coupled with a general interplay (through relational dependency) that is evident between them and in some cases seen for the first time here. In addition, Theorem 2.4 provides a new context for the sequence $\{T_n\}_0^{\infty}$ which is currently absent from [5].

Appendix

We first establish $B(n)$ as in (P.1). Consider

$$\sum_{n=0}^{\infty} B(n)x^n = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} J_i\right)x^n = \frac{G_J(x)}{(1-x)}, \tag{A.1}$$

after a little work. Since

$$\frac{G_J(x)}{(1-x)} = \frac{x}{(1-x)(1+x)(1-2x)} = \frac{1}{2}\left[\frac{1+2x}{(1+x)(1-2x)} - \frac{1}{(1-x)}\right], \tag{A.2}$$

then

$$2\sum_{n=0}^{\infty} B(n)x^n = \frac{1}{(1+x)(1-2x)} + \frac{2x}{(1+x)(1-2x)} - \frac{1}{(1-x)}. \tag{A.3}$$

Now, $1/[(1+x)(1-2x)] = G_J(x)/x = \sum_{n=0}^{\infty} J_n x^{n-1} = \sum_{n=1}^{\infty} J_n x^{n-1} = \sum_{n=0}^{\infty} J_{n+1}x^n$, while $2x/[(1+x)(1-2x)] = 2G_J(x) = 2\sum_{n=0}^{\infty} J_n x^n$ and $1/(1-x) = \sum_{n=0}^{\infty} x^n$, whence (A.3) gives

$$2\sum_{n=0}^{\infty} B(n)x^n = \sum_{n=0}^{\infty}(J_{n+1}+2J_n-1)x^n = \sum_{n=0}^{\infty}(J_{n+2}-1)x^n \tag{A.4}$$

by (L.1), and in turn $B(n) = (J_{n+2}-1)/2$.

Next we derive $A^*(n)$ as in (P.8). Consider

$$\frac{G_J(x/2)}{(1-x)} = (1+x+x^2+x^3+\cdots)\sum_{n=0}^{\infty} J_n(x/2)^n = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{J_i}{2^i}\right)x^n, \tag{A.5}$$

again after some algebraic manipulation. Thus, by the definition of $A^*(n) = \sum_{i=1}^{n} J_{i-1}/2^{i-1}$,

$$\frac{G_J(x/2)}{(1-x)} = \sum_{n=0}^{\infty} A^*(n+1)x^n \tag{A.6}$$

by inspection. Furthermore, it is straightforward to see that $G_J(x/2)/(1-x) = G_S(x/2)$, from which (A.6) delivers

$$\sum_{n=0}^{\infty} A^*(n+1)x^n = G_S(x/2) = \sum_{n=0}^{\infty} S_n(x/2)^n, \tag{A.7}$$

so that $A^*(n+1) = S_n/2^n$, or $A^*(n) = S_{n-1}/2^{n-1}$.

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