On an arithmetic triangle of numbers arising from inverses of analytic functions

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Abstract

The Lagrange inversion formula is a fundamental tool in combinatorics. In this work, we investigate an inversion formula for analytic functions, which does not require taking limits. By applying this formula to certain functions we have found an interesting arithmetic triangle for which we give a recurrence formula. We then explore the links between these numbers, Pascal's triangle, and Bernoulli's numbers, for which we obtain a new explicit formula. Furthermore, we present power series and asymptotic expansions of some elementary and special functions, and some links to the Online Encyclopedia of Integer Sequences (OEIS).

Keywords: inversion formula, analytic functions, arithmetic triangle, recurrent sequences, Bernoulli numbers, integer sequences.

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1 Introduction

The Lagrange inversion formula is a fundamental tool in combinatorics, with numerous applications and many generalizations [14], [6], [7] and [10].

In this work, we introduce a new inversion formula for analytic functions, based on the ordinary one-variable Lagrange inversion. This seems to be both simpler to use, and easier to apply than the classical Lagrange formula, which requires taking limits. Moreover, applying this formula to certain functions we find an arithmetic triangle linked to Pascal's triangle, for which we give a recurrence formula. This continues work presented by the authors in [1].

We also show that these numbers are closely related to Bernoulli numbers, which can now be obtained by a new explicit formula. Furthermore, we present power series and asymptotic expansions of some elementary and special functions [4], involving numbers $\mathcal{A}_n$, and highlight links to existing entries in the Online Encyclopedia of Integer Sequences (OEIS) [13].

2 Inversion formula and applications

2.1 Inversion formula

The inversion of an analytic function

$$u = f(z), \text{ with } z, u \in \mathbb{C},$$

was solved by Lagrange [9] and Bürmann [5] who introduced an inversion formula. The inversion of $f(z)$ is defined as

$$z = g(u) = g(f(z))$$

is given by the Lagrange–Bürmann formula represented as a Taylor series

$$z = g(u) = z_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[\frac{z - z_0}{f(z) - f(z_0)} \right]^k \right) (u - u_0)^k ,$$

where $z_0 = g(u_0)$. For two analytic functions being mutually inverses we have

$$z = g(f(z)), \ z \in \text{Dom}\{f\} \quad \text{and} \quad u = f(g(u)), \ u \in \text{Dom}\{g\}.$$

Differentiating either of them, we get

$$\frac{d}{du} g(u) = \frac{1}{f'(z)}.$$

Applying the differential operator k times, the k-th derivative would be

$$g^{(k)}(u) = \underbrace{\frac{d}{du}\frac{d}{du}\cdots\frac{d}{du}}_{k \text{ times}} g(u) = \frac{d^k}{du^k} g(u) =$$

$$\frac{1}{f'(z)}\left(\frac{d}{dz}\frac{1}{f'(z)}\left(\frac{d}{dz}\frac{1}{f'(z)}\cdots\left(\frac{d}{dz}\frac{1}{f'(z)}\right)\right)\right).$$

By rearranging the brackets, we get

$$g^{(k)}(u) = \left[\frac{1}{f'(z)}\frac{d}{dz}\right]^{k-1}\frac{1}{f'(z)}.$$

The function $f(z)$ is locally univalent in some neighborhood $\mathcal{U}_{z_0}$ of $z_0 \in \mathbb{C}$ due to the condition $f'(z) \neq 0$. Hence, for some $\mathcal{U}_{u_0}$ of $u_0 \in \mathbb{C}$, we have

$$z = g(u) = z_0 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\left[\frac{1}{f'(z_0)}\frac{d}{dz_0}\right]^{k-1}\frac{1}{f'(z_0)}\right)(u-u_0)^k, \tag{1}$$

where $u_0 = f(z_0)$, $z_0 = g(u_0)$. Using the recursion for the coefficients of the series, the formula above can be written more compactly as

$$z = g(u) = z_0 + \sum_{k=1}^{\infty}\frac{1}{k!}\alpha_k(z_0)(u-u_0)^k, \tag{2}$$

where

$$\alpha_1(z_0) = \frac{1}{f'(z_0)} \quad \text{and} \quad \alpha_{k+1}(z_0) = \frac{\alpha_k'(z_0)}{f'(z_0)}.$$

Note that the inverse of a one-to-one function, can be multi-valued. In this case, the points z_0 and u_0 define the branch of the inverse function.

2.2 Applications of the inversion formula

Exp and Log functions. Let us consider two inverse functions

$$u = f(z) = e^z \quad \text{and} \quad z = g(u) = \ln(u).$$

The univalence regions are horizontal strips of 2π-width. Taking $z_0 = 0$ and $u_0 = 1$, by the inversion formula of the exponential function, we find

$$z = g(u) = \sum_{k=1}^{\infty} \frac{1}{k!} \underbrace{e^{-z_0} \frac{d}{dz_0} \dots e^{-z_0} \frac{d}{dz_0} e^{-z_0}}_{k-1 \text{ times}} (u - u_0)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!} e^{-kz_0} (u - u_0)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (u - u_0)^k = \ln u,$$

which is a power series for the principal value of the complex logarithm. The series converges over the interior of a circle at the point $u_0 = 1$, with the radius equal to the distance to the nearest singularity point of $\ln u$, i.e. 0.

Quasi-polynomial. Now we consider the quasi-polynomial

$$u = f(z) = ze^{-\lambda z},$$

with the points $z_0 = 0$ and $u_0 = 0$. We have that

$$f'(z) = (1 - \lambda z)e^{-\lambda z}.$$

Using the recursion formula for the coefficients, we get that

$$\alpha_k(z_0) = \lambda^{k-1} e^{k\lambda z_0} \frac{k^{k-1} + z_0 P_{k-2}(z_0)}{(1 - \lambda z_0)^{2k-1}},$$

where $P_{k-2}(z_0)$ is a polynomial of degree $k-2$. So the inversion formula gives

$$z = g(u) = \sum_{k=1}^{\infty} \frac{(\lambda k)^{k-1}}{k!} u^k.$$

Arctan function. Let us take the arctan function

$$u = f(z) = \arctan z, \quad z = g(u) = \tan u,$$

with the points $z_0 = 0$ and $u_0 = 0$. By the inversion formula, we get

$$z = g(u) = \tan u = \sum_{k=1}^{\infty} \frac{\mathcal{A}_k}{k!} u^k, \quad |u| < \frac{\pi}{2}.$$

The coefficients $\mathcal{A}_k$ are non-negative integers that can be found by

$$\mathcal{A}_k = \left[\left(1 + z_0^2\right) \frac{d}{dz_0} \right]^{k-1} \left(1 + z_0^2\right) \Big|_{z_0=0}, \quad k = 1, 2, \ldots$$

For k even all the coefficients are equal to zero. So, the tan function is a generating function for the numbers $\mathcal{A}_k$.

3 The arithmetic triangle $\mathcal{A}_k$

3.1 Recurrence formula

The numbers $\mathcal{A}_k$ can be defined by means of the recurrence formula. To this end, we construct an analogue of the Pascal's triangle.

We obtain the following table of numbers $a_l^{(k)}$, where k represents the row, while l indicates the column.

$k \setminus l$	1	2	3	4	5	6	7	8	$\mathcal{A}_k$	B_{k+1}
1	1								1	1/6
2	1	1							0	0
3	1	4	1						2	$-1/30$
4	1	11	11	1					0	0
5	1	26	66	26	1				16	1/42
6	1	57	303	302	57	1			0	0
7	1	120	1191	2416	1191	120	1		272	$-1/30$
8	1	247	4293	15619	15619	4293	247	1	0	0

The numerical values corespond to the triangle of Eulerian numbers A008292 in OEIS, which have numerous interpretations. Links to generalized Eulerian numbers have been presented in [2] and [3]. Some combinatorial interpretations of generalised Eulerian numbers are given in [15].

The numbers $a_l^{(k)}$ are computed by the formula

$$a_l^{(k+1)} = l\, a_l^{(k)} + (k - (l-1) + 1)\, a_{l-1}^{(k)}.$$

By using the numbers $a_l^{(k)}$ we can find the numbers $\mathcal{A}_k$ as follows

$$\mathcal{A}_k = \left| \sum_{l=1}^{k} (-1)^l \, a_l^{(k)} \right|, \quad k = 1, 2, \ldots$$

which is the alternating sum of the numbers of the k-th row of the table.

The numbers $\mathcal{A}_k$ have a simple relation with Bernoulli numbers B_k that can be defined explicitly by the formula

$$B_k = \sum_{\nu=0}^{m} \sum_{\mu=0}^{s} (-1)^\mu \binom{s}{\mu} \frac{\mu^m}{s+1},$$

or by generating function as follows

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Namely, we have the following relation between $\mathcal{A}_k$ and B_k.

$$\begin{aligned} B_{2k} &= (-1)^k \frac{2k}{2^{2k}\left(1 - 2^{2k}\right)} \mathcal{A}_{2k-1} \\ &= \frac{2k}{2^{2k}\left(1 - 2^{2k}\right)} \sum_{l=1}^{2k-1} (-1)^l a_l^{(2k-1)}, \;\; k = 1, 2, \ldots. \end{aligned}$$

From the above relation we get the following formula for Bernoulli numbers.

Proposition 3.1 *We have*

$$B_{2k} = (-1)^k \frac{2k}{2^{2k}\left(1 - 2^{2k}\right)} \left[\left(1 + z_0^2\right) \frac{d}{dz_0} \right]^{2k-2} \left(1 + z_0^2\right) \Bigg|_{z_0=0}, \;\; k = 1, 2, \ldots.$$

3.2 Series expansions involving the numbers $\mathcal{A}_k$

We are able to get the series expansions involving the numbers $\mathcal{A}_k$. The expansions of some elementary functions are as follows.

$$\ln \frac{1}{\cos z} = \sum_{k=1}^{\infty} \frac{\mathcal{A}_{2k-1}}{(2k!)} z^{2k}, \qquad |z| < \frac{\pi}{2},$$

which is a generating function for $\mathcal{A}_k$. We obtain the following formulae

$$\frac{z}{\sin z} = 1 + \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)\mathcal{A}_{2k-1}}{2^{2k}\left(2^{2k}-1\right)(2k-1)!} z^{2k}, \qquad |z| < \pi,$$

$$\ln\frac{\sin z}{z} = \sum_{k=1}^{\infty} \frac{\mathcal{A}_{2k-1}}{\left(1-2^{2k}\right)(2k)!} z^{2k}, \qquad |z| < \pi,$$

$$\ln\frac{\tan z}{z} = \sum_{k=1}^{\infty} \frac{\left(2^{2k}-2\right)\mathcal{A}_{2k-1}}{\left(2^{2k}-1\right)(2k-1)!} z^{2k}, \qquad |z| < \frac{\pi}{2},$$

$$z\cot z = 1 + \sum_{k=1}^{\infty} \frac{\mathcal{A}_{2k-1}}{\left(1-2^{2k}\right)(2k-1)!} z^{2k}, \qquad |z| < \pi.$$

For the gamma function we have the following asymptotic series

$$\ln\Gamma(z) = \frac{1}{2}\ln(2\pi) + \left(z-\frac{1}{2}\right)\ln z - z + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\mathcal{A}_{2k-1}}{2^{2k}\left(2^{2k}-1\right)(2k-1)\, z^{2k-1}}$$

The full asymptotic series for the digamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \ln z - \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{\zeta(1-2n)}{z^{2n}} = \ln z - \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n\, z^{2n}}.$$

In terms of the numbers $\mathcal{A}_k$ we have

$$\psi(z) = \ln z - \frac{1}{2z} + \sum_{k=1}^{\infty} \frac{(-1)^k \mathcal{A}_{2k-1}}{2^{2k}\left(2^{2k}-1\right) z^{2k}}.$$

Numbers $\mathcal{A}_k$ can be used in all formulae involving Bernoulli numbers.

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