On some results concerning generalized arithmetic triangles

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Abstract
In this paper we present theoretical and computational results regarding generalized arithmetic \(m\)-triangles. The numerical values recover well-known number sequences, indexed in the OEIS including binomial coefficients and their extensions. Some combinatorial interpretations, generating functions and also asymptotic formulae for these triangles are provided.

Keywords: recurrent sequences, pentanomial numbers, generalized binomial coefficients, asymptotic formulae.
1 Introduction

Let $m$ and $n$ be positive integers. The element of the $m$-arithmetic triangle located at the intersection of the $i$th row and $j$th column denoted by $p_{ij}^{(m)}$ is defined by the recurrence

$$p_{ij}^{(m)} = p_{i-1j}^{(m)} + p_{i-1j-1}^{(m)} + \cdots + p_{i-1j-m+1}^{(m)},$$

with the initial conditions

$$p_{0j}^{(m)} = \begin{cases} 
0 & \text{if } j < 0, \\
1 & \text{if } j = 0, \\
0 & \text{if } j > 0.
\end{cases}$$

The element in each cell is the sum of $m$ elements: the element directly above the given element and the $m - 1$ elements to the left of it. Hence, the matrix

$$P^{(m)}(n) = \left( p_{ij}^{(m)} \right), \quad 0 \leq i, j \leq n - 1$$

of the elements of the $m$-arithmetic triangle is defined as follows

$$p_{ij}^{(m)} = \begin{cases} 
0 & \text{if } i = 0, \ 1 \leq j \leq n - 1, \\
1 & \text{if } j = 0, \ 0 \leq i \leq n - 1 \\
\sum_{l=j-m+1}^{j} p_{i-l}^{(m)} & \text{if } 1 \leq i, j \leq n - 1.
\end{cases}$$

For $m = 2$ one obtains the elements in Pascal’s triangle. Numerous OEIS sequences are obtained from particular columns of the $m$-triangle.

For example, for $p_{n3}^{(3)}$ one obtains the sequence indexed as A005581

$$0, 0, 2, 7, 16, 30, 40, 77, 112, 156, 210, \ldots,$$

in the Online Encyclopedia of Integer Sequences (OEIS) [7].

The sequence $p_{n4}^{(3)}$ whose terms are given by

$$0, 0, 1, 6, 19, 45, 90, 161, 266, 414, 615, \ldots$$

corresponds to sequence A005712.
2 Numerical computation of the $m$-triangles

For a fixed value of $m \geq 2$ and $n \geq 1$ one can compute the rows of the $m$-triangle by matrix iterations. Denoting by $p_i^{(m)}$ the $i$th row of the $m$-triangle, the following formula holds

$$p_{i+1}^{(m)} = p_i^{(m)} M_{m,n},$$

where the matrix $M_{m,n}$ has size $[n(m - 1) + 1] \times [n(m - 1) + 1]$ and has $m$ diagonals whose entries are all equal to 1, as shown below

$$M_{m,n} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 0 & 0 \\
& \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}$$

(1)

One obtains recursively the following identities

$$p_i^{(m)} = p_{i-1}^{(m)} M_{m,n} = p_{i-2}^{(m)} M_{m,n}^2 = \cdots = p_0^{(m)} M_{m,n}^i.$$

Numerous integer sequences are obtained as particular cases:

- $m = 2$: binomial coefficients A007318;
- $m = 3$: trinomial coefficients A027907; The first four rows give the values:
  
  1, 1, 1, 1, 2, 3, 2, 1, 1, 3, 6, 7, 6, 3, 1, 1, 4, 10, 16, 19, 16, 10, 4, 1, \ldots

- $m = 4$: quadrinomial coefficients A008287; First three rows give the values:
  
  1, 1, 1, 1, 1, 2, 3, 4, 3, 2, 1, 1, 3, 6, 10, 12, 12, 10, 6, 3, 1, \ldots

- $m = 5$: pentanomial coefficients A035343.

The sequence of maximum values over each row gives the sequences A001405 ($m = 2$), A002426 ($m = 3$), A035343 ($m = 4$) and A005191 ($m = 5$).

Other related results can be found in [2], [3] and [4].
\section{3 Some Explicit Formulae}

Let $p_{ij}^{(m)}$ be the element located at the intersection of the $i$th row and $j$th column of the $m$-arithmetic triangle. The generating function of these numbers is given by

\[(1 + x + x^2 + \ldots + x^{m-1})^i = \sum_{j=0}^{(m-1)i} p_{ij}^{(m)} x^j, \ m \in \mathbb{N}, \ i \in \mathbb{N} \cup \{0\}.\]

The element $p_{ij}^{(m)}$ is the coefficient of $x^j$ in the formal expansion of

\[(1 + x + x^2 + \ldots + x^{m-1})^i.\]

We can formulate the following results.

\textbf{Theorem 3.1} Let $l = \min\{i, j\}$. Then for $m \in \mathbb{N}$ and $i \in \mathbb{N} \cup \{0\}$

\[p_{ij}^{(m)} = \sum_{s_0 + s_1 + \ldots + s_{m-1} = i \atop s_1 + 2s_2 + \ldots + (m-1)s_{m-1} = j} \frac{l!}{s_0!s_1!\ldots s_{m-1}!}, \ j = 0, 1, \ldots, (m-1)i.\]

\textbf{Theorem 3.2} Let $l = \min\{i, j\}$. Then for $m = 3$ we have

\[p_{ij}^{(3)} = \sum_{k=j-l}^{[j/2]} \frac{i!}{k!(j-2k)!(i+j-k)!}, \ j = 0, \ldots, 2i.\]

\textbf{Example 3.3} As an example, we consider the case when $m = 3$, $i = 4$, $j = 2$. Then, according to Theorem 3.2, $l = 2$, $j - l = 0$, and $[j/2] = 1$, and we get

\[p_{42}^{(3)} = \sum_{k=0}^{1} \frac{4!}{k!(2-2k)!(2+k)!} = \frac{4!}{0!2!2!} + \frac{4!}{1!0!3!} = 10\]

which is exactly the number positioned in the 4th row and 2nd column of the 3-arithmetic triangle (see the Table 1).

\textbf{Example 3.4} Consider the 4-arithmetic triangle and put $m = 4$, $i = 2$, and $j = 3$. Then $l = 2$, and by formula for $p_{ij}^{(m)}$ in Theorem 3.1, we have

\[p_{23}^{(4)} = \sum_{s_0+s_3+s_4+s_3=2 \atop s_1+2s_2+3s_3=3 \atop s_3=0,1,2} \frac{2!}{s_0!s_1!s_2!s_3!}.\]
From the conditions

$$s_1 + 2s_2 + 3s_3 = 3 \quad \text{and} \quad s_1 + s_2 + s_3 \leq 2,$$

we find two sets of solutions

$$\{s_1 = 0, s_2 = 0, s_3 = 1\} \quad \text{and} \quad \{s_1 = 1, s_2 = 1, s_3 = 0\}.$$  

Then the value of $s_0 = 1$ for the first set and $s_0 = 0$ for the second set.

Hence, we obtain $p_{23}^{(4)} = \frac{2!}{1!0!0!0!} + \frac{2!}{0!1!1!0!} = 2+2 = 4$, which is the number positioned at the intersection of the 2th row and 3nd column of the 4-arithmetic triangle (see the Table 2).

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Table 2
4-Arithmetic Triangle
4 Future work

Future investigations will be dedicated to the identification of new integer sequences related to $m$-sequences and to establishing of asymptotic expansions for the numbers $p_{ij}^{(m)}$ when $i, j \rightarrow \infty$ and $m \in \mathbb{N}$, $m \geq 2$ fixed.

The key results in the proof are based on theory given in [5] and [6].

Proposition 4.1 Let $\xi$ be a random variable with the probability distribution

$$ P\{\xi = k\} = \frac{1}{m}, \quad k = 0, 1, \ldots, m - 1. $$

Then the cumulants $c_n$ of a random variable $\xi$ are defined by the formula

$$ c_1 = E[\xi] = \frac{m - 1}{2}, \quad c_{2\nu} = \frac{B_{2\nu}}{2\nu} (m^{2\nu} - 1), \quad c_{2\nu + 1} = 0, $$

where $B_{2\nu}$ are the Bernoulli numbers, $\nu = 1, 2, \ldots$.

Proposition 4.2 Let $\xi_1, \ldots, \xi_i$ be independent random variables with the probability distribution of $\xi$. Then we have

$$ p_{ij}^{(m)} = m^i P\{\xi_1 + \ldots + \xi_i = j\}, \quad j = 0, 1, \ldots, (m - 1)i $$

The formula for $c_n$ above can be derived using the expression [5]

$$ \ln E[e^{\zeta}] = \frac{m - 1}{2}z + \sum_{n=2}^{\infty} \frac{c_n}{n!}z^n, \quad |z| < \frac{2\pi}{m}. $$

Theorem 4.3 Let $i \rightarrow \infty$, $m \geq 2$, $m \in \mathbb{N}$ and let $j \rightarrow \infty$, $j \in \mathbb{N}$, such that

$$ j = \frac{1}{2} \left((m - 1)i + x \sqrt{\frac{i(m^2 - 1)}{3}}\right), \quad |x| \leq c, \quad c = \text{const}. $$

Then, uniformly with respect to $x \in [-c, c]$, we have

$$ p_{ij}^{(m)} = m^i \sqrt{\frac{6}{\pi i(m^2 - 1)}} e^{-\frac{x^2}{2}} \left(1 + \sum_{\nu=1}^{r} \frac{Q_{2\nu}(x)}{i^{\nu}} + O\left(i^{-r-1}\right)\right), \quad r = 1, 2, \ldots $$

where $Q_{2\nu}(x)$ are polynomials in $x$, $\nu = 1, 2, \ldots$, given by

$$ Q_{\nu}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k_1+k_2+\ldots+k_{\nu}=0 \atop k_1+k_2+\ldots+k_{\nu}=s} \prod_{t=1}^{\nu} \frac{1}{k_t!} \frac{\sigma_{t+2}}{(t+2)\sigma_{t+2}}^{k_t}. $$
References


