A Polynomial Based Construction of Periodic Horadam Sequences

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Abstract

A recent matrix based approach to the study of self-repeating Horadam sequences has identified a mechanism to produce guaranteed (and arbitrary) periodicity through a novel formulation for the two governing parameters in the defining linear Horadam recurrence equation. We consider this further here, giving supporting examples to illustrate the methodology which utilises so called Catalan polynomials.

1 Introduction

A Horadam sequence is written $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$, being characterised by the four parameters $a, b, p, q \in \mathbf{C}$ through the defining recurrence

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a, w_1 = b, \tag{1}$$

first introduced and analysed by A.F. Horadam in the 1960s (see the two seminal papers [1,2]). The fully general nature of both the recursion itself and the initial values w_0, w_1 means that many well known, and much considered, (real) sequences can be generated by (1) as special instances, and the literature on linear recurrence equations is vast.

A 2013 survey article [3] constituted a first attempt to set down work carried out on Horadam sequences over almost half a century, one aspect of which has, until recently, been an absence of any proper analysis directed towards their potential to exhibit cyclicity. The focus of this article is the extension of an observation made in [4, Section 3.0.1, p.108] (where a matrix based approach is employed to explore Horadam sequence periodicity and conditions governing it)—namely, the identification of a simple algorithm to generate sequences with any chosen period. Here we show how indeed this works in practice by giving some illustrative examples, adding to the still relatively small number of studies conducted on (linear recurrence) sequence cyclicity (modulo periodicity excepted). Within the methodology adopted, a feature of [4] was the role played throughout by so called Catalan polynomials which are also intrinsic to our generator algorithm.

2 Theory and Results

2.1 Catalan Polynomials and Generator Algorithm

The general $(n + 1)$th Catalan polynomial is written $P_n(x)$. The first few are $P_0(x) = P_1(x) = 1$, $P_2(x) = 1 - x$, $P_3(x) = 1 - 2x$, $P_4(x) = 1 - 3x + x^2$, $P_5(x) = 1 - 4x + 3x^2$, $P_6(x) = 1 - 5x + 6x^2 - x^3$, $P_7(x) = 1 - 6x + 10x^2 - 4x^3, \ldots$, being instances of the closed form

$$P_n(x) = \frac{1}{2^{n+1}} \frac{(1 + \sqrt{1 - 4x}\,)^{n+1} - (1 - \sqrt{1 - 4x}\,)^{n+1}}{\sqrt{1 - 4x}} \tag{2}$$

which holds for $n \geq 0$; $P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i}(-x)^i$ is of degree $\lfloor \frac{1}{2}n \rfloor$.

Let $\delta \geq 3$ be the period of a desired Horadam sequence. Suppose the δth Catalan polynomial $P_{\delta-1}(x)$ has roots $\alpha_1(\delta), \alpha_2(\delta), \ldots, \alpha_{\lfloor \frac{1}{2}(\delta-1) \rfloor}(\delta)$, say. For $r = 1, 2, \ldots, \lfloor \frac{1}{2}(\delta - 1) \rfloor$, any rth root $\alpha_r(\delta)$ gives rise to a non-zero evaluation of the $(\delta + 1)$th Catalan polynomial $P_\delta(\alpha_r(\delta))$.[1] This, in turn, produces δ roots of the equation $x^\delta = P_\delta(\alpha_r(\delta))$ denoted by $d_t^{(r)}(\delta)$ $(t = 1, 2, \ldots, \delta)$, each of which, in conjunction with the generating root $\alpha_r(\delta)$, defines that pair of parameters (present in the recurrence (1)) according to the relations

$$p = p(\delta) = \frac{1}{d_t^{(r)}(\delta)}, \qquad q = q(\delta) = \alpha_r(\delta)\left(\frac{1}{d_t^{(r)}(\delta)}\right)^2, \tag{3}$$

such that the Horadam sequence $\{w_n(a, b; p(\delta), q(\delta))\}_0^\infty$ duly delivered is a period δ one with arbitrary initial values. For a given δ then, from this route, there are clearly a maximum of $\lfloor \frac{1}{2}(\delta - 1) \rfloor \delta$ such sequences available for which the system matrix $\mathbf{A}(p(\delta), q(\delta)) = \left(\begin{smallmatrix} p(\delta) & -q(\delta) \\ 1 & 0 \end{smallmatrix}\right)$ associated with (1) has the property $\mathbf{A}^\delta(p(\delta), q(\delta)) = \mathbf{I}_2$ (the 2×2 identity matrix)—the $p(\delta), q(\delta), \delta$ combination is then said to form what in [4] has been termed an *identity triplet* (and where further details are available), written $[p, q, \delta]$.

2.2 Examples and Results

Example 1: $\delta = 3$ The sole root of $P_{3-1}(x) = P_2(x) = 1 - x$ is $\alpha_1(3) = 1$. We look, therefore, for the three cube roots of $P_3(\alpha_1(3)) = P_3(1) = -1$, which are $d_1^{(1)}(3) = -1$, $d_2^{(1)}(3) = \frac{1}{2}(1 + \sqrt{3}i)$ and $d_3^{(1)}(3) = \frac{1}{2}(1 - \sqrt{3}i)$. These, by (3), generate the three p, q pairs

$$p = \frac{1}{d_1^{(1)}(3)} = -1, \quad q = \frac{\alpha_1(3)}{\left(d_1^{(1)}(3)\right)^2} = 1,$$

$$p = \frac{1}{d_2^{(1)}(3)} = \frac{1}{2}(1 - \sqrt{3}i), \quad q = \frac{\alpha_1(3)}{\left(d_2^{(1)}(3)\right)^2} = -\frac{1}{2}(1 + \sqrt{3}i),$$

[1] Note that, importantly, *no* root $\alpha_r(\delta)$ of the δth Catalan polynomial $P_{\delta-1}(x)$ can also be a root of the $(\delta + 1)$th polynomial $P_\delta(x)$ since it is known that [5, Theorem 2, p.42] for $n \geq 2$ any consecutive triplet of Catalan polynomials $P_{n-1}(x), P_n(x), P_{n+1}(x)$ has pairwise distinct sets of roots; thus, if $P_{\delta-1}(\alpha_r(\delta)) = 0$ it follows that $P_\delta(\alpha_r(\delta)) \neq 0$.

$$p = \frac{1}{d_3^{(1)}(3)} = \frac{1}{2}(1+\sqrt{3}i), \quad q = \frac{\alpha_1(3)}{\left(d_3^{(1)}(3)\right)^2} = \frac{1}{2}(-1+\sqrt{3}i), \quad (4)$$

with associated identity triplets $[-1,1,3], [\frac{1}{2}(1-\sqrt{3}i), -\frac{1}{2}(1+\sqrt{3}i), 3]$ and $[\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), 3]$, and (respective) sequences

$$\begin{aligned}
\{w_n(a,b;-1,1)\}_0^\infty &= \{a, b, -(a+b), \ldots\}, \\
\left\{w_n\left(a,b;\frac{1}{2}(1-\sqrt{3}i), -\frac{1}{2}(1+\sqrt{3}i)\right)\right\}_0^\infty &= \\
&\left\{a, b, \frac{1}{2}(a+b) + \frac{\sqrt{3}}{2}(a-b)i, \ldots\right\}, \\
\left\{w_n\left(a,b;\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i)\right)\right\}_0^\infty &= \\
&\left\{a, b, \frac{1}{2}(a+b) + \frac{\sqrt{3}}{2}(b-a)i, \ldots\right\}, \quad (5)
\end{aligned}$$

of period 3.

Example 2: $\delta = 4$ This time we have but one root of $P_{4-1}(x) = P_3(x) = 1-2x$ to generate a sequence set, which is $\alpha_1(4) = \frac{1}{2}$. The four quartic roots of $P_4(\alpha_1(4)) = P_4(\frac{1}{2}) = -\frac{1}{4}$ are $d_1^{(1)}(4) = \frac{1}{2}(1+i)$, $d_2^{(1)}(4) = \frac{1}{2}(-1+i)$, $d_3^{(1)}(4) = -\frac{1}{2}(1+i)$ and $d_4^{(1)}(4) = \frac{1}{2}(1-i)$, with equations (3) yielding p, q pairs

$$\begin{aligned}
p &= \frac{1}{d_1^{(1)}(4)} = 1-i, \quad q = \frac{\alpha_1(4)}{\left(d_1^{(1)}(4)\right)^2} = -i, \\
p &= \frac{1}{d_2^{(1)}(4)} = -(1+i), \quad q = \frac{\alpha_1(4)}{\left(d_2^{(1)}(4)\right)^2} = i, \\
p &= \frac{1}{d_3^{(1)}(4)} = -1+i, \quad q = \frac{\alpha_1(4)}{\left(d_3^{(1)}(4)\right)^2} = -i, \\
p &= \frac{1}{d_4^{(1)}(4)} = 1+i, \quad q = \frac{\alpha_1(4)}{\left(d_4^{(1)}(4)\right)^2} = i, \qquad (6)
\end{aligned}$$

associated identity triplets $[1-i, -i, 4], [-(1+i), i, 4], [-1+i, -i, 4], [1+i, i, 4]$, and corresponding period 4 sequences

$$\{w_n(a,b;1-i,-i)\}_0^\infty = \{a, b, b+(a-b)i, a+(a-b)i, \ldots\},$$

$$
\begin{aligned}
\{w_n(a,b;-(1+i),i)\}_0^\infty &= \{a,b,-b-(a+b)i,-a+(a+b)i,\ldots\},\\
\{w_n(a,b;-1+i,-i)\}_0^\infty &= \{a,b,-b+(a+b)i,-a-(a+b)i,\ldots\},\\
\{w_n(a,b;1+i,i)\}_0^\infty &= \{a,b,b+(b-a)i,a+(b-a)i,\ldots\}. \qquad (7)
\end{aligned}
$$

Example 3: $\delta = 6$ The Catalan polynomial $P_{6-1}(x) = P_5(x) = 1 - 4x + 3x^2 = (1-x)(1-3x)$ has two roots $\alpha_1(6) = 1$ and $\alpha_2(6) = \frac{1}{3}$. The first root, through the six roots of $P_6(\alpha_1(6)) = 1$, generates a set of six p, q pairs and period 6 sequences, as does the second root through the six roots of $P_6(\alpha_2(6)) = -\frac{1}{27}$. The mechanism for p, q pair construction having been established above, it suffices to merely list the identity triplets $[1,1,6]$, $[\frac{1}{2}(1-\sqrt{3}i), -\frac{1}{2}(1+\sqrt{3}i), 6]$, $[-\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), 6]$, $[-1,1,6]$, $[\frac{1}{2}(-1+\sqrt{3}i), -\frac{1}{2}(1+\sqrt{3}i), 6]$, $[\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), 6]$ associated with $\alpha_1(6)$ (of which the 2nd, 4th and 6th are implicit from Example 1, being period 6 because they are in fact period 3), and those triplets $[\frac{3}{2}(1-\frac{1}{\sqrt{3}}i), \frac{1}{2}(1-\sqrt{3}i), 6]$, $[-\sqrt{3}i, -1, 6]$, $[-\frac{3}{2}(1+\frac{1}{\sqrt{3}}i), \frac{1}{2}(1+\sqrt{3}i), 6]$, $[\frac{3}{2}(-1+\frac{1}{\sqrt{3}}i), \frac{1}{2}(1-\sqrt{3}i), 6]$, $[\sqrt{3}i, -1, 6]$, $[\frac{3}{2}(1+\frac{1}{\sqrt{3}}i), \frac{1}{2}(1+\sqrt{3}i), 6]$ associated with $\alpha_2(6)$; representative sequence examples are

$$
\begin{aligned}
&\{w_n(a,b;1,1)\}_0^\infty = \{a,b,b-a,-a,-b,-(b-a),\ldots\},\\
&\left\{w_n\left(a,b;\frac{1}{2}(-1+\sqrt{3}i),-\frac{1}{2}(1+\sqrt{3}i)\right)\right\}_0^\infty =\\
&\qquad\left\{a,b,\frac{1}{2}(a-b)+\frac{\sqrt{3}}{2}(a+b)i,-a,-b,-\frac{1}{2}(a-b)-\frac{\sqrt{3}}{2}(a+b)i,\ldots\right\},\\
&\{w_n(a,b;\sqrt{3}i,-1)\}_0^\infty =\\
&\qquad\{a,b,a+\sqrt{3}bi,-2b+\sqrt{3}ai,-2a-\sqrt{3}bi,b-\sqrt{3}ai,\ldots\},\\
&\left\{w_n\left(a,b;\frac{3}{2}(1+\frac{1}{\sqrt{3}}i),\frac{1}{2}(1+\sqrt{3}i)\right)\right\}_0^\infty =\\
&\qquad\left\{a,b,\frac{1}{2}(3b-a)+\frac{\sqrt{3}}{2}(b-a)i,b+\sqrt{3}(b-a)i,\right.\\
&\qquad\qquad\left.a+\sqrt{3}(b-a)i,\frac{1}{2}(3a-b)+\frac{\sqrt{3}}{2}(b-a)i,\ldots\right\}. \qquad (8)
\end{aligned}
$$

Example 4: $\delta = 8$ The three roots of $P_7(x) = 1 - 6x + 10x^2 - 4x^3 = (1-2x)(1-4x+2x^2)$ are all real, being $\alpha_1(8) = \frac{1}{2}$, $\alpha_{2,3}(8) = 1 \pm \frac{1}{\sqrt{2}}$. Associated with $\alpha_1(8)$ (through the eight roots of $P_8(\alpha_1(8))$ are eight triplets $[\pm\sqrt{2},1,8]$, $[\pm(1-i),-i,8]$,$[\pm\sqrt{2}i,-1,8]$,$[\pm(1+i),i,8]$, and corresponding sequences such as $\{w_n(a,b;-\sqrt{2},1)\}_0^\infty = \{a,b,-(a+\sqrt{2}b),\sqrt{2}a+b,-a,-b,a+\sqrt{2}b,-(\sqrt{2}a+b),\ldots\}$ and $\{w_n(a,b;\sqrt{2}i,-1)\}_0^\infty = \{a,b,a+\sqrt{2}bi,\sqrt{2}ai-b,-a,-b,-(a+\sqrt{2}bi),-(\sqrt{2}ai-b),\ldots\}$ (four of which are period 4 (and so period 8), having appeared in Example 2). Two further sets (of eight) of fully period 8 sequences are delivered by $\alpha_{2,3}(8)$, each of which has been verified computationally.

We finish with a final, and somewhat more exotic, example to illustrate the robustness of the construction algorithm presented.

Example 5: $\delta = 23$ For $\delta = 23$ there are 11 roots of the (non-reducible) polynomial $P_{22}(x) = 1-21x+190x^2-969x^3+3060x^4-6188x^5+8008x^6-6435x^7+3003x^8-715x^9+66x^{10}-x^{11}$, being (to 4 d.p.) $\alpha_1(23) = 0.2547$, $\alpha_2(23) = 0.2696$, $\alpha_3(23) = 0.2972$, $\alpha_4(23) = 0.3425$, $\alpha_5(23) = 0.4155$, $\alpha_6(23) = 0.5366$, $\alpha_7(23) = 0.7517$, $\alpha_8(23) = 1.1811$, $\alpha_9(23) = 2.2293$, $\alpha_{10}(23) = 6.0395$ and $\alpha_{11}(23) = 53.6823$.

The first root $\alpha_1(23)$, via the 23 roots $d_1^{(1)}(23),\ldots,d_{23}^{(1)}(23)$ of $P_{23}(\alpha_1(23)) = O(10^{-7})$,[2] produces 23 identity triplets, the first of which is $[1.962917287347780-0.2697967711570217i, 0.9629172873477802-0.2697967711570191i, 23] = [1/d_1^{(1)}(23), \alpha_1(23)/(d_1^{(1)}(23))^2, 23]$ and gives rise to the period 23 sequence (again with 4 d.p. accuracy)

$$
\begin{aligned}
w_0 &= a,\\
w_1 &= b,\\
w_2 &= 1.9629b-0.9629a+(0.2698a-0.2698b)i,\\
w_3 &= 2.8173b-1.8173a+(0.7894a-0.7894b)i,\\
w_4 &= 3.4999b-2.4999a+(1.5202a-1.5202b)i,\\
w_5 &= 3.9600b-2.9600a+(2.4081a-2.4081b)i,\\
w_6 &= 4.1634b-3.1634a+(3.3872a-3.3872b)i,
\end{aligned}
$$

[2] All bar one of these roots are complex, which collectively lie equally spaced on a complex disk of radius 0.5047008105853039. As the (real) roots $\alpha_1(23),\ldots,\alpha_{11}(23)$ increase in size, so do the magnitudes of $P_{23}(\alpha_1(23)),\ldots,P_{23}(\alpha_{11}(23))$ (on which sequences are based in sets of 23) ranging from $O(10^{-7})$ to $O(10^{19})$.

$$
\begin{aligned}
w_7 &= 4.0952b - 3.0952a + (4.3849a - 4.3849b)i,\\
w_8 &= 3.7603b - 2.7603a + (5.3271a - 5.3271b)i,\\
w_9 &= 3.1836b - 2.1836a + (6.1441a - 6.1441b)i,\\
w_{10} &= 2.4079b - 1.4079a + (6.7752a - 6.7752b)i,\\
w_{11} &= 1.4907b - 0.4907a + (7.1736a - 7.1736b)i,\\
w_{12} &= 0.5000b + 0.5000a + (7.3097a - 7.3097b)i,\\
w_{13} &= -0.4907b + 1.4907a + (7.1736a - 7.1736b)i,\\
w_{14} &= -1.4079b + 2.4079a + (6.7752a - 6.7752b)i,\\
w_{15} &= -2.1836b + 3.1836a + (6.1441a - 6.1441b)i,\\
w_{16} &= -2.7603b + 3.7603a + (5.3271a - 5.3271b)i,\\
w_{17} &= -3.0952b + 4.0952a + (4.3849a - 4.3849b)i,\\
w_{18} &= -3.1634b + 4.1634a + (3.3872a - 3.3872b)i,\\
w_{19} &= -2.9600b + 3.9600a + (2.4081a - 2.4081b)i,\\
w_{20} &= -2.4999b + 3.4999a + (1.5202a - 1.5202b)i,\\
w_{21} &= -1.8173b + 2.8173a + (0.7894a - 0.7894b)i,\\
w_{22} &= -0.9629b + 1.9629a + (0.2698a - 0.2698b)i, \qquad (9)
\end{aligned}
$$

etc. The other 22 identity triplets associated with this same root $\alpha_1(23)$ are simply $[1/d_t^{(1)}(23), \alpha_1(23)/(d_t^{(1)}(23))^2, 23]$ $(t = 2, \ldots, 23)$, each the architect of its own sequence. The whole scenario is repeated for the remaining roots $\alpha_2(23), \ldots, \alpha_{11}(23)$, with a total of $11 \times 23 = 253$ sequences generated; all have been computer checked which, given the range of sizes of $P_{23}(\alpha_1(23)), \ldots, P_{23}(\alpha_{11}(23))$ indicated in Footnote 2, is a pleasing endorsement of the methodology presented.

3 Summary

In this paper we have looked at a simple and novel algorithm—featuring the use of Catalan polynomials and illustrated by examples—which enables the construction of (arbitrary initial values) Horadam sequences with any designated period, adding a small layer of knowledge to our understanding of the self-repeating potential of these sequences. The unusual nature of the method is striking, and one wonders if an enhanced (or equivalent) version might exist so as to facilitate the identification of cyclic sequences associated with a linear recurrence equation of order three or more—we leave this, for the moment, as an open problem.

Notwithstanding the fact that, for a given δ, any divisor(s) of δ reduce(s) the number of unique period δ sequences from the maximum $\lfloor \frac{1}{2}(\delta-1) \rfloor \delta$ created (since some will of course have smaller period(s)), there are clearly a considerable number of periodic Horadam sequences delivered using this mechanism (the sequence $\{\lfloor \frac{1}{2}(\delta-1) \rfloor \delta\}_{\delta=3}^{\infty} = \{3, 4, 10, 12, 21, 24, 36, 40, 55, 60, \ldots\}$ is an increasing one with quadratic growth, its terms being those coefficients of $x^0, x^1, x^2, x^3, \ldots$ in the power series expansion of $(3+x)/(1+x)^2(1-x)^3$ which acts as a generating function for the sequence; see O.E.I.S. Sequence No. A050187 for a linear order 5 recurrence satisfied by these terms). Moreover, the work here contrasts greatly with other approaches where periodicity is determined by basic properties of roots of unity and their relationship with the governing parameters p, q of (1) (see [6,7] for characterisations of sequence behaviours, and [8] for a particular type of sequence trait described as 'masked' periodicity). We mention, too, some additional related work that might be of interest to the reader: a more detailed enumerative analysis of possible cyclic Horadam sequences already exists [9], and Catalan polynomials have been seen to underpin periodic Horadam sequences of a certain type in [10]. We are of the opinion that self-repeating Horadam sequences might be examined using methods which run along yet other lines of enquiry, as the topic seems to be a mathematically rich one and offers scope for further analysis.

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