On Horadam Sequence Periodicity: A New Approach

Peter J. Larcombe and Eric J. Fennessey[†]

School of Computing and Mathematics
University of Derby
Kedleston Road, Derby DE22 1GB, U.K.
{P.J.Larcombe@derby.ac.uk}

[†]BAE Systems Integrated System Technologies
Broad Oak, The Airport, Portsmouth PO3 5PQ, U.K.
{Eric.Fennessey@baesystems.com}

Abstract

A so called Horadam sequence is one delivered by a general second order recurrence formula with arbitrary initial conditions. We examine aspects of self-repeating Horadam sequences by applying matrix based methods in new ways, and derive some conditions governing their cyclic behaviour. The analysis allows for both real and complex sequence periodicity.

1 Introduction

Consider the sequence $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$ defined by the linear recurrence

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a, w_1 = b, \tag{1}$$

of order 2, particular values of p, q, a, b giving rise to some well known sequences (Fibonacci, Pell, Lucas, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Tagiuri, Fermat, Fermat-Lucas, for example). We call this general sequence a Horadam sequence, based on the studies of A.F. Horadam begun in the 1960s (although some authors regard (1) as a generalised Fibonacci/Lucas recurrence as it gives rise to the namesake sequences $\{w_n(0,1;1,-1)\}_0^\infty$ and $\{w_n(2,1;1,-1)\}_0^\infty$, resp.). Note also that both types of Chebyshev polynomial—$T_n(x)$ (of the first kind) and $U_n(x)$ (of the second kind)—are solutions of (1) when $p = 2x$ and $q = 1$, with (for $n \geq 0$) $T_n(x) = w_n(1,x;2x,1)$ and $U_n(x) = w_n(1,2x;2x,1)$.

As remarked by Larcombe and Fennessey in [1]—the forerunner to this article—some isolated work on instances of sequences generated by (1) had occurred during the late 19th century and early part of the 20th century, but it was Horadam who made the first serious study of this type of fully general recursion, creating an initial suite of results (in two 1965 publications) to provide himself and others with motivation for further research at that time and for many years to come. It was in [1] that the discovery of three Horadam sequence instances was reported, of form $\{w_n(1,\sqrt{s};\sqrt{s},1)\}_0^\infty$, which exhibit cyclicity of period 6, 8 and 12 for respective values of $s = 1, 2$ and 3; it was found that

$$\begin{aligned}
\{w_n(1,1;1,1)\}_0^\infty &= \{1,1,0,-1,-1,0,\ldots\} \\
&= \{P_n(1)\}_0^\infty, \\
\{w_n(1,\sqrt{2};\sqrt{2},1)\}_0^\infty &= \{1,\sqrt{2},1,0,-1,-\sqrt{2},-1,0,\ldots\} \\
&= \{(\sqrt{2})^n P_n(1/2)\}_0^\infty, \\
\{w_n(1,\sqrt{3};\sqrt{3},1)\}_0^\infty &= \{1,\sqrt{3},2,\sqrt{3},1,0,-1,-\sqrt{3},-2,-\sqrt{3},-1,0,\ldots\} \\
&= \{(\sqrt{3})^n P_n(1/3)\}_0^\infty, \qquad (2)
\end{aligned}$$

where $P_n(x)$ is the $(n+1)$th so called Catalan polynomial. These sequences were identified in [1] as those fundamental periodic ones arising from analysis of the closed form

$$P_n(x) = \frac{1}{2^{n+1}} \frac{(1+\sqrt{1-4x}\,)^{n+1} - (1-\sqrt{1-4x}\,)^{n+1}}{\sqrt{1-4x}} \qquad (3)$$

for $P_n(x)$ given in [2, (70), p.17], the paper in which the polynomials are first reported. The existence of these sequences was both surprising and pleasing, especially as Horadam has only ever once, it seems, briefly mentioned the period 3 and 6 sequences [3, (2.35),(2.36), p.166]

$$\begin{aligned}
\{w_n(a,b;-1,1)\}_0^\infty &= \{a,b,-(a+b),\ldots\}, \\
\{w_n(a,b;1,1)\}_0^\infty &= \{a,b,b-a,-a,-b,-(b-a),\ldots\}, \qquad (4)
\end{aligned}$$

with the latter, for $a = b = 1$, seen in (2) (see also Remark 1). The stand out feature of what follows is the integral role which is played by Catalan polynomials in our analysis.

In this paper we pursue the notion of Horadam periodicity using matrix based methods, formulating governing conditions accordingly and providing examples throughout. The characteristic equation $0 = \lambda^2 - p\lambda + q$ associated with (1) has non-degenerate ($p^2 \neq 4q$) and degenerate ($p^2 = 4q$) case roots potentially, and both are discussed in the following section as we bring to bear our approach to the topic. The underlying ideas are developed further in Section 3 where other aspects of periodicity are considered and a hitherto unseen phenomenon identified which we term 'masked' periodicity. A Summary concludes the presentation, preceded by some general remarks.

2 Periodicity Conditions and Analysis

2.1 Non-Degenerate Case ($p^2 \neq 4q$)

Let w_0, w_1 be the arbitrary a, b initial values of a general Horadam sequence $\{w_n(a, b; p, q)\}_0^\infty$ (where $a, b, p, q \in \mathbf{C}$). Writing the recurrence (1) as

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{n-1} \\ w_{n-2} \end{pmatrix} \tag{5}$$

in matrix form leads readily to the matrix power equation

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}, \tag{6}$$

which holds for $n \geq 1$. We are interested in the condition(s) under which the sequence $\{w_n\}_0^\infty$ is cyclic, with (minimal) period $\delta \geq 1$, say, so that $w_n = w_{n+\delta}$ ($n \geq 0$), noting that for $\delta = 1$ then $\{w_n\}_0^\infty = \{w_0, w_0, w_0, \ldots\} = \{a, a, a, \ldots\}$ can be generated by the collapsed $p = 1$, $q = 0$ form of (1) (*i.e.*, $w_n = w_{n-1}$) read as an order 1 recursion with initial value $w_0 = a$ (see also Section 2.2). Using (6), therefore (and assuming $p, q \neq 0$), we impose periodicity as

$$\begin{aligned} \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n+\delta-1} \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} &= \begin{pmatrix} w_{n+\delta} \\ w_{n+\delta-1} \end{pmatrix} \\ &= \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}. \end{aligned} \tag{7}$$

In other words, writing

$$\mathbf{A} = \mathbf{A}(p, q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{w} \;=\; \mathbf{w}(w_0, w_1) \;=\; (w_1, w_0)^T, \tag{8}$$

then, reconciling both sides of (7),

$$\mathbf{A}^\delta \mathbf{w} \;=\; \mathbf{w}, \tag{9}$$

which means that $\mathbf{w}$ is an eigenvector of $\mathbf{A}^\delta$ corresponding to an eigenvalue of unity. Fortunately, we can evaluate the matrix $\mathbf{A}^\delta$ by applying an unusual, but useful, identity from Clapperton *et al.* [4, (21), p.142], namely,

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n \;=\; \begin{pmatrix} P_n(-xy) & xP_{n-1}(-xy) \\ yP_{n-1}(-xy) & xyP_{n-2}(-xy) \end{pmatrix}, \tag{10}$$

which gives, with $x = -q/p$, $y = 1/p$,

$$\begin{aligned} \mathbf{A}^\delta(p,q) &= p^\delta \begin{pmatrix} 1 & -q/p \\ 1/p & 0 \end{pmatrix}^\delta \\ &= p^\delta \begin{pmatrix} P_\delta(q/p^2) & -\frac{q}{p}P_{\delta-1}(q/p^2) \\ \frac{1}{p}P_{\delta-1}(q/p^2) & -\frac{q}{p^2}P_{\delta-2}(q/p^2) \end{pmatrix} \\ &= \begin{pmatrix} p^\delta \rho_\delta & -qp^{\delta-1}\rho_{\delta-1} \\ p^{\delta-1}\rho_{\delta-1} & -qp^{\delta-2}\rho_{\delta-2} \end{pmatrix} \end{aligned} \tag{11}$$

in terms of Catalan polynomials, where $\rho_\delta(p,q) = P_\delta(q/p^2)$. The characteristic equation we require to obtain is

$$\begin{aligned} 0 &= \begin{vmatrix} p^\delta \rho_\delta - \lambda & -qp^{\delta-1}\rho_{\delta-1} \\ p^{\delta-1}\rho_{\delta-1} & -(qp^{\delta-2}\rho_{\delta-2} + \lambda) \end{vmatrix} \\ &= \lambda^2 - p^{\delta-2}(p^2\rho_\delta - q\rho_{\delta-2})\lambda + qp^{2(\delta-1)}(\rho_{\delta-1}^2 - \rho_\delta\rho_{\delta-2}) \end{aligned} \tag{12}$$

after a little algebra. Thus, $\lambda = 1$ guarantees periodicity when

$$\begin{aligned} 0 &= 1 - p^{\delta-2}(p^2\rho_\delta - q\rho_{\delta-2}) + qp^{2(\delta-1)}(\rho_{\delta-1}^2 - \rho_\delta\rho_{\delta-2}) \\ &= 1 - p^{\delta-2}(p^2\rho_\delta - q\rho_{\delta-2}) + q^\delta, \end{aligned} \tag{13}$$

the final r.h.s. term having been simplified by employing the Catalan polynomial result (a derivation of which is given in Appendix A)

$$x^n \;=\; P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \tag{14}$$

with $n = \delta - 1$, $x = q/p^2$. Thus, *a general necessary (but not sufficient) condition for periodicity* is, for $\delta \geq 2$,

$$F(p,q,\delta) \;=\; 1 + p^\delta \left(\frac{q}{p^2} P_{\delta-2}(q/p^2) - P_\delta(q/p^2) \right) + q^\delta \;=\; 0, \tag{15}$$

which we note is independent of the sequence starting values w_0, w_1 (the cyclic nature of sequences being characterised by the defining recursion parameters p, q).

2.1.1 Specific Examples

As stated earlier, the three periodic sequences of (2) are each a consequence of the Horadam recurrence scheme

$$w_n = \sqrt{s}\,w_{n-1} - w_{n-2}; \qquad w_0 = 1, \;\; w_1 = \sqrt{s}, \tag{16}$$

for which $p = \sqrt{s}$ $(s = 1, 2, 3)$, $q = 1$. Setting $q = 1$, therefore, Horadam periodicity such as we have observed in [1] should occur from (15) for $F(p, 1, \delta) = 0$, where

$$F(p, 1, \delta) = 2 + p^{\delta}\left(\frac{1}{p^2}P_{\delta-2}(1/p^2) - P_{\delta}(1/p^2)\right), \tag{17}$$

and we anticipate the three instances $0 = F(1, 1, 6) = F(\sqrt{2}, 1, 8) = F(\sqrt{3}, 1, 12)$. We leave it as a reader excercise to verify that these identities hold (using the Catalan polynomial listing of Appendix B), and we note that since the 2×2 identity matrix $\mathbf{I}_2$ has a (repeated) eigenvalue of 1, then the three $p, q = 1, \delta$ combinations seen could describe special cases of the result $\mathbf{A}^{\delta}(p, 1) = \mathbf{I}_2$ (*i.e*, $\mathbf{A}^6(1, 1) = \mathbf{A}^8(\sqrt{2}, 1) = \mathbf{A}^{12}(\sqrt{3}, 1) = \mathbf{I}_2$) for which (9) is satisfied trivially for arbitrary $\mathbf{w}$; in fact computations confirm

$$\mathbf{I}_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^6 = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}^8 = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & 0 \end{pmatrix}^{12}. \tag{18}$$

The condition (15) is such that a cyclic Horadam sequence parameterised by p, q, and with period δ, will automatically satisfy it.

<u>Remark 1</u> We know that the period 3 Horadam sequence $\{w_n(a, b; -1, 1)\}_0^{\infty}$ given in (4) should satisfy the implied condition (17), which is to say $F(-1, 1, 3) = 0$. Noting that $P_1(x) = 1$, $P_3(x) = 1 - 2x$, we can check this easily: $F(-1, 1, 3) = 2 + (-1)^3[1 \cdot P_1(1) - P_3(1)] = 2 - [P_1(1) - P_3(1)] = 2 - [1 - (-1)] = 0$. It duly yields its own matrix identity, namely,

$$\mathbf{A}^3(-1, 1) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^3 = \mathbf{I}_2. \tag{19}$$

With reference to those sequences in (2) it does not seem possible to identify a closed form for $w_n(a, b; -1, 1)$ which is expressible in terms of the general Catalan polynomial $P_n(x)$, hence its omission from [1] where these three sequences were a direct result of analysis of the polynomials.

<u>Remark 2</u> The non-degenerate ($p^2 \neq 4q$) and degenerate ($p^2 = 4q$) root cases are always treated independently, often appealing to the respective

analytic solutions of the recurrence (1). If $p^2 \neq 4q$ the roots $\alpha = \frac{1}{2}(p + \sqrt{p^2 - 4q})$, $\beta = \frac{1}{2}(p - \sqrt{p^2 - 4q})$, say, of the associated characteristic equation are distinct, and the general sequence term is easily written in a closed (Binet) form $w_n = [(b - a\beta)\alpha^n - (b - a\alpha)\beta^n]/(\alpha - \beta)$ $(n \geq 0)$. The degenerate case solution is $w_n = bn(\frac{1}{2}p)^{n-1} - a(n-1)(\frac{1}{2}p)^n$, for which the characteristic equation has equal roots $\alpha = \beta = \frac{1}{2}p$. We do not utilise either form in our methodology, though it is of interest to see what (15) gives in the instance of degeneracy. We have necessarily, for a period δ sequence,

$$\begin{aligned} 0 &= F(p, p^2/4, \delta) \\ &= 1 + p^\delta \left(\frac{1}{4} P_{\delta-2}(1/4) - P_\delta(1/4) \right) + \left(\frac{p^2}{4} \right)^\delta \\ &= 1 + p^\delta \left(\frac{1}{4} \frac{(\delta - 1)}{2^{\delta-2}} - \frac{(\delta + 1)}{2^\delta} \right) + \left(\frac{p}{2} \right)^{2\delta} \end{aligned} \tag{20}$$

using the result $P_n(1/4) = (n+1)/2^n$ $(n \geq 0)$ established in [2, Remark 2, p.18]. This then reduces to the simple condition $0 = [(\frac{1}{2}p)^\delta - 1]^2$, which is to say that $\frac{1}{2}p$ must be a δth root of unity; we will see this condition re-appear shortly as we formulate separate conditions for periodicity in the presence of degeneracy.

2.1.2 A Theorem

To finish this subsection we give a theorem. Having established the necessary condition $0 = F(p, q, \delta)$ (15) for periodicity, we can also obtain a sufficiency argument since working the algebra backwards from (15) to (13) we are able to re-arrange the latter as

$$qp^{2(\delta-1)}(\rho_{\delta-1}^2 - \rho_\delta \rho_{\delta-2}) = p^{\delta-2}(p^2 \rho_\delta - q\rho_{\delta-2}) - 1, \tag{21}$$

whereupon the characteristic equation (12) reads, writing for convenience $\gamma = \gamma(p, q, \delta) = p^{\delta-2}(p^2 \rho_\delta - q\rho_{\delta-2})$,

$$\begin{aligned} 0 &= \lambda^2 - \gamma\lambda + \gamma - 1 \\ &= (\lambda - 1)(\lambda - \gamma + 1). \end{aligned} \tag{22}$$

We see that $\mathbf{A}^\delta(p, q)$ has an eigenvalue of 1 (as well as $\gamma - 1$), and taking $\mathbf{w} = (w_1, w_0)^T$ as the corresponding eigenvector of $\mathbf{A}^\delta$ then $\{w_n(w_0, w_1; p, q)\}_0^\infty$ will be a cyclic sequence. Thus, we have our result:

Theorem 1 *For fixed p, q, δ $(p^2 \neq 4q)$ there exists a non-zero vector $\mathbf{w} = (w_1, w_0)^T$ such that $\mathbf{A}^\delta \mathbf{w} = \mathbf{w}$ iff $F(p, q, \delta) = 0$; the resulting sequence $\{w_n(w_0, w_1; p, q)\}_0^\infty$ is a cyclic one of period δ.*

2.2 Degenerate Case ($p^2 = 4q$)

Here we modify our approach slightly from the non-degenerate case above, taking advantage of the closed form available for $P_n(1/4)$. We will show that, when $p^2 = 4q$, *then* $(\frac{1}{2}p)^\delta = 1$ *and* $w_1 = (\frac{1}{2}p)w_0$ *are together necessary and sufficient conditions for periodicity*, and we furnish our findings with some useful sequence exemplars for clarity.

To establish necessity we assume periodicity, that is to say (9), where

$$\mathbf{A} \quad = \quad \mathbf{A}(p) \quad = \quad \begin{pmatrix} p & -p^2/4 \\ 1 & 0 \end{pmatrix}, \tag{23}$$

with (11) now reading

$$\begin{aligned} \mathbf{A}^\delta(p) \quad &= \quad \begin{pmatrix} p^\delta P_\delta(\frac{1}{4}) & -\frac{1}{4}p^{\delta+1}P_{\delta-1}(\frac{1}{4}) \\ p^{\delta-1}P_{\delta-1}(\frac{1}{4}) & -\frac{1}{4}p^\delta P_{\delta-2}(\frac{1}{4}) \end{pmatrix} \\ &= \quad \begin{pmatrix} (\delta+1)(\frac{1}{2}p)^\delta & -\delta(\frac{1}{2}p)^{\delta+1} \\ \delta(\frac{1}{2}p)^{\delta-1} & -(\delta-1)(\frac{1}{2}p)^\delta \end{pmatrix}. \end{aligned} \tag{24}$$

Writing (9) as

$$\mathbf{A}^\delta \mathbf{w} - \mathbf{w} \quad = \quad \mathbf{0} \tag{25}$$

gives component equations

$$0 \quad = \quad [(\delta+1)(p/2)^\delta - 1]w_1 - \delta(p/2)^{\delta+1}w_0, \tag{26}$$

and

$$0 \quad = \quad \delta(p/2)^{\delta-1}w_1 - [(\delta-1)(p/2)^\delta + 1]w_0, \tag{27}$$

from which latter equation

$$w_1 \quad = \quad \frac{(\delta-1)(p/2)^\delta + 1}{\delta(p/2)^{\delta-1}}w_0 \tag{28}$$

and in turn, by back-substitution into (26),

$$[(\delta+1)(p/2)^\delta - 1] \cdot \frac{(\delta-1)(p/2)^\delta + 1}{\delta(p/2)^{\delta-1}}w_0 \quad = \quad \delta(p/2)^{\delta+1}w_0. \tag{29}$$

Simplification of (29) (assuming $w_0 \neq 0$, for we would otherwise have the trivial zero sequence for which $w_0 = w_1 = w_2 = w_3 = \cdots = 0$ from (28),(1)) reduces it to merely $0 = [(\frac{1}{2}p)^\delta - 1]^2$, or $(\frac{1}{2}p)^\delta = 1$, with (28) itself then contracting to the simple relation $w_1 = (\frac{1}{2}p)w_0$.

Sufficiency is shown under the assumption that $(\frac{1}{2}p)^\delta = 1$ and $w_1 = (\frac{1}{2}p)w_0$. We write, from (23),(24),

$$\begin{aligned} \mathbf{A}^\delta \mathbf{w} &= \begin{pmatrix} p & -p^2/4 \\ 1 & 0 \end{pmatrix}^\delta \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\ &= \begin{pmatrix} (\delta+1)(\frac{1}{2}p)^\delta & -\delta(\frac{1}{2}p)^{\delta+1} \\ \delta(\frac{1}{2}p)^{\delta-1} & -(\delta-1)(\frac{1}{2}p)^\delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}, \end{aligned} \tag{30}$$

and the assumptions now enable us to yield the required criteria for periodicity:

$$\begin{aligned} \mathbf{A}^\delta \mathbf{w} &= \begin{pmatrix} \delta+1 & -\delta(\frac{1}{2}p) \\ \delta/(\frac{1}{2}p) & -(\delta-1) \end{pmatrix} \begin{pmatrix} (\frac{1}{2}p)w_0 \\ w_0 \end{pmatrix} \\ &= \begin{pmatrix} (\frac{1}{2}p)w_0 \\ w_0 \end{pmatrix} \\ &= \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\ &= \mathbf{w}. \end{aligned} \tag{31}$$

We feel it useful to illustrate that *both* of the stated conditions are required to guarantee periodicicity of the Horadam sequence arising from the degenerate case recursion

$$w_n = pw_{n-1} - (p^2/4)w_{n-2}; \tag{32}$$

we give a couple of examples, and then discuss further aspects of the degenerate case.

Example 1

I: We choose $p = 2$, δ (arbitrary) ≥ 1, so that $(\frac{1}{2}p)^\delta = 1$. The sequence given by (32) is $\{w_n(a, b; 2, 1)\}_0^\infty = \{a, b, 2b-a, 3b-2a, 4b-3a, 5b-4a, 6b-5a, \ldots\}$ which is, as expected, not periodic. However, if the further condition $b = w_1 = (\frac{1}{2}p)w_0 = w_0 = a$ is imposed, then this sequence becomes $\{w_n(a, a; 2, 1)\}_0^\infty = \{a, a, a, a, a, a, \ldots\}$ which is indeed cyclic with (minimal) period 1.

II: This time, if we first take $b = w_1 = (\frac{1}{2}p)w_0 = \frac{1}{2}pa$ alone we obtain the non-periodic sequence $\{w_n(a, \frac{1}{2}pa; p, \frac{1}{4}p^2)\}_0^\infty = \{a, \frac{1}{2}pa, \frac{1}{4}p^2a, \frac{1}{8}p^3a, \frac{1}{16}p^4a, \frac{1}{32}p^5a, \frac{1}{64}p^6a, \ldots\}$; this re-forms the periodic sequence $\{w_n(a, a; 2, 1)\}_0^\infty$ on subsequently imposing $p = 2$.

Example 2

I: Here we choose $p = -2$, δ (even) ≥ 2, with $(\frac{1}{2}p)^\delta = 1$ holding. The

recurrence formula this time yields the (non-periodic) sequence $\{w_n(a,b;-2,1)\}_0^\infty = \{a, b, -(a+2b), 2a+3b, -(3a+4b), 4a+5b, -(5a+6b), \ldots\}$. If $b = w_1 = (\frac{1}{2}p)w_0 = -w_0 = -a$ additionally, then this sequence becomes the predicted periodic one (of minimal even period 2) $\{w_n(a,-a;-2,1)\}_0^\infty = \{a, -a, a, -a, a, -a, \ldots\}$.
II: With $b = w_1 = (\frac{1}{2}p)w_0 = \frac{1}{2}pa$ set initially, the non-periodic sequence delivered by (32) is $\{w_n(a, \frac{1}{2}pa; p, \frac{1}{4}p^2)\}_0^\infty$ seen in Example 1. Setting further $p = -2$, this reads as the periodic sequence $\{w_n(a,-a;-2,1)\}_0^\infty$.

There is a complete class of sequences within this degenerate case, for suppose $r^{(\delta)}$ is a primitive δth root of unity ($\delta \geq 1$). Then, if $p = 2r^{(\delta)}$ and (given $w_0 = a$) $b = w_1 = r^{(\delta)}a$, both periodicity conditions are satisfied and we anticipate that the sequence $\{w_n(a, r^{(\delta)}a; 2r^{(\delta)}, (r^{(\delta)})^2)\}_0^\infty$ will have minimal period δ. In fact it is easy to see this, for (32) reads in this instance $w_n = 2r^{(\delta)}w_{n-1} - (r^{(\delta)})^2 w_{n-2}$ and generates the sequence with general $(n+1)$th term $w_n = (r^{(\delta)})^n a$. Thus we can write

$$\begin{aligned}
&\{w_n(a, r^{(\delta)}a; 2r^{(\delta)}, (r^{(\delta)})^2)\}_0^\infty \\
&\quad = \{(r^{(\delta)})^n a\}_0^\infty \\
&\quad = \{a, r^{(\delta)}a, (r^{(\delta)})^2 a, \ldots, (r^{(\delta)})^{\delta-1}a, (r^{(\delta)})^{\delta}a, (r^{(\delta)})^{\delta+1}a, (r^{(\delta)})^{\delta+2}a, \ldots\} \\
&\quad = \{a, r^{(\delta)}a, (r^{(\delta)})^2 a, \ldots, (r^{(\delta)})^{\delta-1}a, (r^{(\delta)})^{\delta}a, \\
&\qquad\qquad (r^{(\delta)})^{\delta} r^{(\delta)}a, (r^{(\delta)})^{\delta}(r^{(\delta)})^2 a, \ldots\} \\
&\quad = \{a, r^{(\delta)}a, (r^{(\delta)})^2 a, \ldots, (r^{(\delta)})^{\delta-1}a, a, r^{(\delta)}a, (r^{(\delta)})^2 a, \ldots\}; \qquad (33)
\end{aligned}$$

the sequence terms are located around the radius a circle in the complex plane and describe repeat circuits as the sequence progresses. By way of illustration, denoting the two complex primitive cube roots of unity as $r_1^{(3)} = \frac{1}{2}(-1+\sqrt{3}i)$, $r_2^{(3)} = \frac{1}{2}(-1-\sqrt{3}i)$, we find that

$$\begin{aligned}
\{w_n(a, r_1^{(3)}a; 2r_1^{(3)}, (r_1^{(3)})^2 = r_2^{(3)}\}_0^\infty &= \{a, r_1^{(3)}a, r_2^{(3)}a, \ldots\}, \\
\{w_n(a, r_2^{(3)}a; 2r_2^{(3)}, (r_2^{(3)})^2 = r_1^{(3)}\}_0^\infty &= \{a, r_2^{(3)}a, r_1^{(3)}a, \ldots\}, \qquad (34)
\end{aligned}$$

are (minimal) period 3 sequences, and the two complex primitive fourth roots $\pm i$ of unity yield (minimal) period 4 sequences

$$\begin{aligned}
\{w_n(a, ai; 2i, -1)\}_0^\infty &= \{a, ai, -a, -ai, \ldots\}, \\
\{w_n(a, -ai; -2i, -1)\}_0^\infty &= \{a, -ai, -a, ai, \ldots\}. \qquad (35)
\end{aligned}$$

For those real roots of unity in particular, then (a) $r^{(\delta)} = 1$ has an associated sequence $\{w_n(a,a;2,1)\}_0^\infty = \{a, a, a, a, a, a, \ldots\}$ of (minimal) period 1, whilst (b) $r^{(\delta)} = -1$ has an associated sequence $\{w_n(a,-a;-2,1)\}_0^\infty =$

$\{a, -a, a, -a, a, -a, \ldots\}$ of (minimal) period 2; these roots are primitive roots of (resp.) degree 1,2, and produce precisely the sequences seen in Examples 1,2 above. It is obvious that any non-primitive δth root of unity will result in a sequence whose minimal period is a divisor of δ.

3 Further Analysis

Consider the matrix $\mathbf{A} = \mathbf{A}(p, q)$ (8). We define $[p, q, \delta]$ to be an *identity triplet* if $\mathbf{A}^\delta(p, q) = \mathbf{I}_2$, and we have already seen four such triplets $[1, 1, 6], [\sqrt{2}, 1, 8], [\sqrt{3}, 1, 12], [-1, 1, 3]$ in (18),(19). We prove the following.

Lemma *Suppose $p^2 \neq 4q$. Then $[p, q, \delta]$ is an identity triplet if and only if (i) $P_{\delta-1}(q/p^2) = 0$ and (ii) $P_\delta(q/p^2) = 1/p^\delta$.*

Proof We prove only the necessary part of the result since the sufficient part is a straightforward reverse argument. Suppose $[p, q, \delta]$ is an identity triplet so that, using (11),

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \mathbf{I}_2 \\ &= \mathbf{A}^\delta(p, q) \\ &= \begin{pmatrix} p^\delta P_\delta(q/p^2) & -qp^{\delta-1}P_{\delta-1}(q/p^2) \\ p^{\delta-1}P_{\delta-1}(q/p^2) & -qp^{\delta-2}P_{\delta-2}(q/p^2) \end{pmatrix}. \qquad (36) \end{aligned}$$

On reconciling matrix terms across (36) it is immediate that $P_{\delta-1}(q/p^2) = 0$, which is Condition (i). We must also have $p^\delta P_\delta(q/p^2) = 1$, that is to say Condition (ii), which latter ensures that $-qp^{\delta-2}P_{\delta-2}(q/p^2) = 1$ as required (this is a simple consequence of evaluating the known linear Catalan polynomial recurrence [2, (69), p.17] $P_n(x) = P_{n-1}(x) - xP_{n-2}(x)$ at $x = q/p^2$, for then it gives $P_\delta(q/p^2) = P_{\delta-1}(q/p^2) - (q/p^2)P_{\delta-2}(q/p^2) \Rightarrow 1/p^\delta = 0 - (q/p^2)P_{\delta-2}(q/p^2) \Rightarrow 1 = -qp^{\delta-2}P_{\delta-2}(q/p^2)$).□

Thus, for instance, the four identity triplets identified each have the following easily checked n.a.s. conditions:

$[1, 1, 6]$: (i) $P_5(1) = 0$ and (ii) $P_6(1) = 1$;
$[\sqrt{2}, 1, 8]$: (i) $P_7(\frac{1}{2}) = 0$ and (ii) $16P_8(\frac{1}{2}) = 1$;
$[\sqrt{3}, 1, 12]$: (i) $P_{11}(\frac{1}{3}) = 0$ and (ii) $729P_{12}(\frac{1}{3}) = 1$;
$[-1, 1, 3]$: (i) $P_2(1) = 0$ and (ii) $-P_3(1) = 1$,

and we have arrived, with reference to (9), at a new formal statement for periodicity:

Theorem 2 *Suppose $p^2 \neq 4q$. If $[p,q,\delta]$ form an identity triplet, the sequence $\{w_n(a,b;p,q)\}_0^\infty$ is periodic with period δ.*

Theorem 2 is different in nature from Theorem 1, and it does not appear possible to deduce one from the other.

3.0.1 Construction of a Periodic Sequence

One way to construct a sequence such that Conditions (i),(ii) are satisfied by choice of p,q,δ is by setting $p=[P_\delta(\alpha)]^{-1/\delta}$ and $q=\alpha[P_\delta(\alpha)]^{-2/\delta}$, where $\alpha \neq 0$ is a root of the Catalan polynomial $P_{\delta-1}(x)$.[1] Then, since $q/p^2=\alpha$ we can write $0=P_{\delta-1}(\alpha)=P_{\delta-1}(q/p^2)$ (Condition (i)) and (noting that if $P_{\delta-1}(\alpha)=0$ then[2] $P_\delta(\alpha)\neq 0$) we also have $p^\delta=[P_\delta(\alpha)]^{-1}=[P_\delta(q/p^2)]^{-1}$, or $P_\delta(q/p^2)=1/p^\delta$ (Condition (ii)); it duly follows that $\mathbf{A}^\delta(p,q)=\mathbf{I}_2$ (using an argument along the same lines as in the proof of the Lemma). However, having fixed both $\delta \geq 3$ (since $P_2(x)$ is the first polynomial to possess a root), and the root $x=\alpha$ of $P_{\delta-1}(x)$, then in order to *guarantee* a suitable choice of the pair p,q it suffices to first set $p=p(\alpha;\delta)=[P_\delta(\alpha)]^{-1/\delta}$, and then set $q=q(\alpha;\delta)=q(\alpha,p(\alpha;\delta))=\alpha p^2$.

3.1 Other Examples of Periodicity

Moving on, we emphasise that Theorem 2 is only a sufficient condition for periodicity, and there will exist many cases where particular initial conditions, in conjunction with choice of p,q,δ, give rise to cyclic (minimal period δ) sequences for which $\mathbf{A}^\delta(p,q)\neq \mathbf{I}_2$; we give three examples which have been checked computationally:

Example A: $p=1$, $q=-2$, $\delta=2$ $(p,q\in\mathbf{R})$
First we note that

$$\mathbf{A}^2(1,-2) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \neq \mathbf{I}_2. \qquad (37)$$

The sequence $\{w_n(a,b;1,-2)\}_0^\infty = \{a,b,2a+b,2a+3b,6a+5b,10a+11b,\ 22a+21b,\ldots\}$ becomes the period 2 sequence $\{w_n(a,-a;1,-2)\}_0^\infty =$

[1]Roots of the general Catalan polynomial $P_n(x)$ are non-zero, being given by [2, (97), p.25] $x=1/\{4\cos^2[\lambda\pi/(n+1)]\}$, where λ takes values $\lambda=1,2,\ldots,n$ (excluding any value for which $2\lambda=n+1$).

[2]We need to invoke [5, Theorem 2, p.42] which states that for $n\geq 2$ any consecutive triplet of Catalan polynomials $P_{n-1}(x),P_n(x),P_{n+1}(x)$ has pairwise distinct sets of roots.

$\{a, -a, \ldots\}$ for $b = w_1 = -a$, and we see that the vector $\mathbf{w} = (w_1, w_0)^T = (b, a)^T = (-a, a)^T$ is an eigenvector of $\mathbf{A}^2$ with eigenvalue 1 (*i.e.*, $\mathbf{A}^2\mathbf{w} = \mathbf{w}$); this is, of course, a requirement of (9).

Example B: $p = \frac{1}{2}(-3 + \sqrt{3}i)$, $q = 2$, $\delta = 3$ ($p \in \mathbf{C}$, $q \in \mathbf{R}$)
Here,

$$\begin{aligned} \mathbf{A}^3\left(\frac{1}{2}(-3+\sqrt{3}i), 2\right) &= \begin{pmatrix} \frac{1}{2}(-3+\sqrt{3}i) & -2 \\ 1 & 0 \end{pmatrix}^3 \\ &= \begin{pmatrix} 6+\sqrt{3}i & 1+3\sqrt{3}i \\ -\frac{1}{2}(1+3\sqrt{3}i) & 3-\sqrt{3}i \end{pmatrix} \\ &\neq \mathbf{I}_2. \end{aligned} \tag{38}$$

The non-periodic sequence $\{w_n(a, b; \frac{1}{2}(-3 + \sqrt{3}i), 2)\}_0^\infty$ becomes, with $b = w_1 = -\frac{1}{2}(1 + \sqrt{3}i)a$, the period 3 sequence $\{w_n(a, -\frac{1}{2}(1 + \sqrt{3}i)a; \frac{1}{2}(-3 + \sqrt{3}i), 2)\}_0^\infty = \{a, -\frac{1}{2}(1 + \sqrt{3}i)a, -\frac{1}{2}(1 - \sqrt{3}i)a, \ldots\}$, and again we observe $\mathbf{w} = (w_1, w_0)^T$ is an eigenvector of $\mathbf{A}^3$ with eigenvalue 1.

Example C: $p = -1 + 2i$, $q = -(1 + i)$, $\delta = 4$ ($p, q \in \mathbf{C}$)
We have

$$\begin{aligned} \mathbf{A}^4(-1+2i, -(1+i)) &= \begin{pmatrix} -1+2i & 1+i \\ 1 & 0 \end{pmatrix}^4 \\ &= \begin{pmatrix} -4+5i & 5+5i \\ 5 & 1-5i \end{pmatrix} \\ &\neq \mathbf{I}_2. \end{aligned} \tag{39}$$

The non-cyclic sequence $\{w_n(a, b; -1 + 2i, -(1 + i))\}_0^\infty$ becomes a period 4 sequence on choosing $b = w_1 = ai$, with $\{w_n(a, ai; -1 + 2i, -(1 + i))\}_0^\infty = \{a, ai, -a, -ai, \ldots\}$; it is once more readily checked that $\mathbf{w} = (w_1, w_0)^T$ is a (unit eigenvalue) eigenvector of $\mathbf{A}^4$.

By Theorem 1 we expect (15) to be satisfied for each p, q, δ combination in Examples A-C (the resulting periodic sequences being initial conditions dependent), and some simple computational checks confirm that $0 = F(1, -2, 2) = F(\frac{1}{2}(-3 + \sqrt{3}i), 2, 3) = F(-1 + 2i, -(1 + i), 4)$.

3.1.1 Masked Periodicity

We finish Section 3 with a final example which exposes yet one more aspect of periodicity which we term *masked* periodicity. From those we have identified, we have chosen a non-trivial one to include here.

Example D: $p=-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i$, $q=\frac{1}{\sqrt{2}}(1+i)$ $(p,q\in\mathbf{C})$

Noting that

$$\begin{aligned}\mathbf{A}^4\left(-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i,\frac{1}{\sqrt{2}}(1+i)\right) &= \begin{pmatrix}(1-\sqrt{2})i & \sqrt{2}-1-i\\ \sqrt{2}-1+i & (\sqrt{2}-1)i\end{pmatrix}\\ &\neq \mathbf{I}_2, \qquad (40)\end{aligned}$$

then, as in Examples A-C, we can find an eigenvector $\mathbf{w}=(a,ai)^T$ such that $\mathbf{A}^4\mathbf{w}=\mathbf{w}$, the sequence

$$\begin{aligned}&\left\{w_n\left(ai,a;-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i,\frac{1}{\sqrt{2}}(1+i)\right)\right\}_0^\infty\\ &\qquad = \{ai,a,-ai,-a,\ldots\} \qquad (41)\end{aligned}$$

has period 4, and

$$F\left(-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i,\frac{1}{\sqrt{2}}(1+i),4\right) = 0. \qquad (42)$$

However in this instance we have, additionally, the matrix power identity

$$\begin{aligned}&\mathbf{A}^8\left(-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i,\frac{1}{\sqrt{2}}(1+i)\right)\\ &\qquad = \begin{pmatrix}(1-\sqrt{2})i & \sqrt{2}-1-i\\ \sqrt{2}-1+i & (\sqrt{2}-1)i\end{pmatrix}^2\\ &\qquad = \mathbf{I}_2, \qquad (43)\end{aligned}$$

and an identity triplet $[-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i,\frac{1}{\sqrt{2}}(1+i),8]$ for which *arbitrary* initial values return the period 8 sequence $\{w_n(a,b;-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i,\frac{1}{\sqrt{2}}(1+i))\}_0^\infty$ with repeating terms

$$\begin{aligned}w_0 &= a,\\ w_1 &= b,\\ w_2 &= -\frac{1}{\sqrt{2}}(a+b)+\left[\left(\frac{1}{\sqrt{2}}-1\right)b-\frac{1}{\sqrt{2}}a\right]i,\\ w_3 &= \left(1-\frac{1}{\sqrt{2}}\right)a+\left(\frac{1}{\sqrt{2}}-1\right)b+\left[\frac{1}{\sqrt{2}}a+\left(\frac{1}{\sqrt{2}}-1\right)b\right]i,\\ w_4 &= (\sqrt{2}-1)b+[(\sqrt{2}-1)a+b]i,\\ w_5 &= (\sqrt{2}-1)a+[(1-\sqrt{2})b-a]i,\\ w_6 &= \left(\frac{1}{\sqrt{2}}-1\right)a+\left(1-\frac{1}{\sqrt{2}}\right)b+\left[\left(1-\frac{1}{\sqrt{2}}\right)a-\frac{1}{\sqrt{2}}b\right]i,\\ w_7 &= -\frac{1}{\sqrt{2}}(a+b)+\left[\left(1-\frac{1}{\sqrt{2}}\right)a+\frac{1}{\sqrt{2}}b\right]i, \qquad (44)\end{aligned}$$

etc., and *for which the period 4 sequence (41) is recovered when $a \to ai$, $b = a$.* Thus, we have a specific (in terms of designated initial values) 'higher frequency/smaller period' sequence hidden within, or *masked by*, a general (arbitrary initial values) 'lower frequency/larger period' one. In this there is a tangential link with some degenerate case sequences where, as we have observed, if $\frac{1}{2}p$ is a δth root of unity then non-periodic ones become cyclic upon applying a particular initial condition constraint (namely, $w_1 = (p/2)w_0$); here we remain periodic, but move to a higher frequency. This is an interesting trait, and there are other examples of the phenomenon of masked periodicity (Appendix C) which collectively, perhaps, offer a separate topic for study in order to better understand its basic characteristics.

Remark 3 Note that the two Lemma conditions

$$\begin{aligned} P_7\left(\frac{\frac{1}{\sqrt{2}}(1+i)}{\left[-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i\right]^2}\right) &= 0, \\ P_8\left(\frac{\frac{1}{\sqrt{2}}(1+i)}{\left[-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i\right]^2}\right) &= \frac{1}{\left[-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i\right]^8}, \end{aligned} \tag{45}$$

associated with the identity triplet $[-\frac{1}{\sqrt{2}}+(\frac{1}{\sqrt{2}}-1)i, \frac{1}{\sqrt{2}}(1+i), 8]$ have been verified computationally.

The Catalan Polynomial Sequences

We end—for the purpose of completeness—with some remarks on the notion of masked periodicity in relation to the sequences discussed at the opening of the paper. The recurrence (16), with the particular starting values shown, produces the period $6, 8, 12$ sequences $\{w_n(1, \sqrt{s}; \sqrt{s}, 1)\}_0^\infty$ for respective values of $s = 1, 2, 3$, as we have seen in (2). Sequences of the same period, however, are generated from *arbitrary* starting values,[3] that is to say,

$$\begin{aligned} \{w_n(a,b;1,1)\}_0^\infty &= \{a, b, b-a, -a, -b, -(b-a), \ldots\}, \\ \{w_n(a,b;\sqrt{2},1)\}_0^\infty &= \{a, b, \sqrt{2}b-a, b-\sqrt{2}a, \\ &\qquad -a, -b, -(\sqrt{2}b-a), -(b-\sqrt{2}a), \ldots\}, \\ \{w_n(a,b;\sqrt{3},1)\}_0^\infty &= \\ &\{a, b, \sqrt{3}b-a, 2b-\sqrt{3}a, \sqrt{3}b-2a, b-\sqrt{3}a, \\ &\quad -a, -b, -(\sqrt{3}b-a), -(2b-\sqrt{3}a), -(\sqrt{3}b-2a), -(b-\sqrt{3}a), \ldots\}. \end{aligned} \tag{46}$$

[3] This is no surprise in view of the identity triplets $[1,1,6], [\sqrt{2},1,8], [\sqrt{3},1,12]$ highlighted at the start of Section 3.

At first sight each has the capacity to mask a smaller period (higher frequency), initial value specific sequence. This is not manifest, however, as the powers of $\mathbf{A}(\sqrt{s},1)$ shown in (18) are, for $s = 1,2,3$, in each case the *lowest* one to result in the identity matrix $\mathbf{I}_2$, with no intermediate (smaller) power ever possessing a unit eigenvalue;[4] we can, therefore, say that the Catalan polynomial sequences $\{P_n(1)\}_0^\infty, \{(\sqrt{2})^n P_n(1/2)\}_0^\infty$ and $\{(\sqrt{3})^n P_n(1/3)\}_0^\infty$, with which we began our presentation, are merely initial value instances of more general, same period, sequences (46).

4 Some Further Remarks

4.1 On Certain Horadam Sequence Closed Forms

As a small technical embellishment of the final paragraph of the opening section of the paper, we make some remarks (for the interested reader) on certain Horadam sequences which—based on their closed forms—are real ones.

For $p,q \in \mathbf{R}$ the recurrence (1) will yield a real Horadam sequence $\{w_n(a,b;p,q)\}_0^\infty$ for real initial values $a = w_0$, $b = w_1$. With reference to the Introduction earlier, the building blocks, so to speak, of a closed form for the general $(n+1)$th term w_n are the characteristic roots α, β of the characteristic equation of (1). Whilst in the degenerate case ($p^2 = 4q$; $\alpha(p) = \beta(p) = \frac{1}{2}p$) the closed form $w_n(p,a,b) = bn(\frac{1}{2}p)^{n-1} - a(n-1)(\frac{1}{2}p)^n$ is self-evidently real, it is not immediately obvious that the same can be said by consideration of the closed form $w_n(\alpha,\beta,a,b) = [(b-a\beta)\alpha^n - (b-a\alpha)\beta^n]/(\alpha-\beta)$ for non-degeneracy ($p^2 \neq 4q$; $\alpha(p,q) = \frac{1}{2}(p+\sqrt{p^2-4q})$, $\beta(p,q) = \frac{1}{2}(p-\sqrt{p^2-4q})$) when $p^2 < 4q$, for then $\alpha(p,q), \beta(p,q)$ form a complex conjugate pair;[5] this is the case for each of the sequences of (2), for instance. To see why, in fact, w_n is real (containing, as it does, integral powers of both α, β), we obtain an alternative form for it comprising only real (Polar) constituent components (and a,b)—a process which proves instructive. Writing $p^2 - 4q = -\Delta^2$ (where $\Delta^2(p,q) = 4q - p^2 > 0$) gives a Cartesian form $\alpha(p,q) = \alpha_r + i\alpha_c$, where $\alpha_r(p) = \mathrm{Re}\{\alpha\} = \frac{1}{2}p$, $\alpha_c(p,q) = \mathrm{Im}\{\alpha\} = \frac{1}{2}\Delta$ ($\alpha_r, \alpha_c \in \mathbf{R}$); correspondingly, $\beta(p,q) = \alpha_r - i\alpha_c$. It then follows that $\alpha, \beta = r\exp[\pm i\theta] = r[\cos(\theta) \pm i\sin(\theta)]$ for some $\theta(p,q) = \theta(\alpha_r(p), \alpha_c(p,q))$, $r(q)$ (with $r^2(q) = q$). Using de Moivre's Theorem $\alpha^n, \beta^n = r^n[\cos(n\theta) \pm i\sin(n\theta)]$ trivially and, after some tedious but elementary algebraic manipulation, it is found that $w_n(r,\theta,a,b) = \{b\sin(n\theta) - ar\sin[(n-1)\theta]\}r^{n-1}/\sin(\theta)$.

[4] For the same reason (with reference to (19)), Horadam's 1965 period 3 sequence $\{w_n(a,b;-1,1)\}_0^\infty = \{a,b,-(a+b),\ldots\}$ of (4) masks no other sequence either.

[5] Obviously, if $p^2 > 4q$ then distinct $\alpha(p,q), \beta(p,q) \in \mathbf{R}$ and so too is $w_n(\alpha,\beta,a,b)$.

4.2 Other Remarks

4.2.1 Horadam Periodicity

We emphasise that although Horadam himself made one (incidental) remark in 1965 on the two minor periodic realisations (4) of his sequences, it would seem (see our 2013 survey work [6]) that the idea of Horadam sequence cyclicity lay dormant in the literature over many years until articles [7,8] to which this presentation provides an additional viewpoint. Clearly, whilst Horadam sequences continued to be the subject of interest from the mid 1960s onwards, the possibility that they might possess periodic aspects was—deliberately or otherwise—ignored (bar some articles on modulo periodicity, a somewhat different type of behaviour) as researchers looked to other types of problems on which to work within the field of linear order two recurrence equations.

4.2.2 Other Periodicity Contexts

On a slightly more general point we first note—purely for the purpose of completeness—that particular types of *non-linear* recursive equations have been, and still are, of some interest in the context of periodicity since some of the techniques employed in their analysis can be applied to the investigation of models arising in many areas (such as, for example, biology, engineering, economics, sociology and physics). Whilst unrelated to our work, so called max-type (and, to a lesser extent, min-type) difference equations are among those non-linear systems which have occasioned scrutiny, in part for those (albeit mathematically abstract) periodic characteristics they may possess;[6] seemingly an emergent area of research in the 1990s, a good starting point for access to literature—in terms of background information and references—are the 2009/10 papers by Elsayed and Iričanin [9,10] (although further publications have, of course, appeared since then).

Whilst prime divisors of second order recurrence sequences are related to their modulo periodic behaviour, modulo type properties of Horadam terms set themselves aside as results which do not impinge on our work, although a paper from the 1970s worth mentioning is by Alter and Kubota [11] who looked at the interesting problem of Horadam sequence multiplicities for p, q integer (given integer k the multiplicity of the Horadam sequence $m(k)$ is the number of solutions of $k = w_n$), bringing together previous work on second order recurrence multiplicity, p-adic theory and congruence class theory from the 1950s and 1960s; obviously, the type of periodic sequences

[6]Cyclicity is also to be found in the solutions of certain piecewise linear systems as well.

considered by us have infinite multiplicity.

An unrelated, though nonetheless interesting, article on periodic second order linear recurrence sequences to highlight is due to McGuire [12] who has examined the composition of Fibonacci type recursions (based on Horadam parameters $p = \pm 1$, $q = -1$) generated by plus and minus 'rules' applied to two chosen (starting) 'seeds', giving necessary conditions for a sequence to be periodic and showing all possible periods of such sequences. The work has echoes of ours to the extent that modulus one eigenvalues of the so called motif matrix (absorbing the plus/minus string combinations) are linked to sequence period. A rather different, and very loosely connected, paper which involves periodic aspects of some order two integer recurrence sequences through mappings is that by Puri and Ward [13], whilst Lau [14] considered a recurrence which itself displays periodicity.

4.2.3 Other Horadam Related Work

Isolated congruence results involving combinations of Horadam terms appeared in the 1990s [15, Theorems 2,3, pp.426-427; 16, Theorems 4.1,4.2, pp.267-268; 17, Theorem 3, p.328], and the congruence properties of second order recurrence sequence terms are obviously connected to the notion of sequence term divisibility. Questions on sequence zeros (modulo an integer) were studied many years ago. Hall [18] gave a necessary and sufficient criterion for the existence (modulo a prime) of Horadam integer sequences zeros, improving on a less direct (and a little more restrictive) analysis by Ward [19] (who much later showed, for example, that prime modulo zeros exist for an infinite number of primes [20]). Important papers such as these, and others around at that time, fostered in this field an initial momentum which has been sustained to a certain extent. It was not included in the survey [6], however, since its appeal lies more with number theoreticians than with those whose concern is the world of number sequences where Horadam and one or two contemporaries first produced results and gave Horadam sequences a research foothold. We should also mention that up until the early 1990s it was the case that among the considerable number of studies on prime divisors of integer sequence terms, those that examined the natural density of divisor sets were relatively few. Motivated by this, Ballot drew together work from four mathematicians to produce a trilogy of results [21] which dealt with non-degenerate (in the sense that the ratio of any two characteristic roots is not a root of unity) linear recurrence sequences of any order whose terms are integer; concepts and methods of Lucas, Laxton, Hasse and Lagarias relating to second order linear recurrences and prime division were necessarily reviewed and generalised in his timely and useful publication which has since served as a major reference for research in this area.

5 Summary

In this paper we have attempted to examine, using matrix based methodologies, parameters governing the existence and behaviour of real/complex Horadam sequences which exhibit periodicity; necessary and sufficient conditions have been given for both non-degenerate and degenerate characteristic root cases of the Horadam recurrence, together with other results and observations (see also Appendix D). An obvious aspect of the analysis is the role, through (10), of the Catalan polynomials, which continues the theme of the work in [1] from which this paper follows in consequence. Another feature is the contrast between our approach here and that in [7,8] where Horadam sequence characteristic roots and general term closed forms underpin the analysis of sequences for which a, b, p, q are assumed complex, and a characterisation of periodic orbits (and some non-periodic motions) can be found;[7] certainly these works, combined with the present one, mean that the mechanics and various facets of Horadam cyclicity are now beginning to be understood in some depth for the first time.

To end, it is emphasised that the appearance of masked periodic sequences has given rise to what might prove to be a fruitful line of future enquiry, as potentially does the cyclic sequence construction algorithm outlined in Section 3.0.1. Insofar as these two topics are identified as such, then clearly the manner in which Horadam periodicity has been tackled in this paper—and the insights gained in the process—are together without doubt mathematically propitious.

6 Acknowledgement

The constructive comments of the referee, to whom we are grateful, have improved the layout and readability of the paper in places.

[7]The results in [7,8] are formulated using the initial conditions a, b and, more importantly, so called generators which absorb the Horadam characteristic roots (dependent on p, q) and offer a suitable form of the sequence general term closed form on which to base analysis. There is but little overlap between the findings of this study and [7] (the n.a.s. conditions of the degenerate case here are encapsulated in Theorems 3.3,3.5 therein (pp.33,34)), which latter also includes the identification of inner and outer boundaries for regions in the complex plane containing periodic orbits. It is possible to determine all essential types of closed paths, and further examples of geometric configurations (such as polygons and bipartite graphs) are presented in [8].

Appendix A

The derivation of (14) (deployed in Section 2.1) is given for completeness.

Consider the known linear recurrence [2, (69), p.17]

$$0 = xP_n(x) - P_{n+1}(x) + P_{n+2}(x), \tag{A1}$$

or

$$xP_{n-1}(x) = P_n(x) - P_{n+1}(x). \tag{A2}$$

To obtain (14) we use a non-linear recurrence

$$x^n = P_n^2(x) + xP_{n-1}^2(x) - P_n(x)P_{n-1}(x) \tag{A3}$$

(which made only a brief appearance in [22, (6), p.44]), using (A2) to write

$$\begin{aligned} x^n &= P_n^2(x) + [xP_{n-1}(x)]P_{n-1}(x) - P_n(x)P_{n-1}(x) \\ &= P_n^2(x) + [P_n(x) - P_{n+1}(x)]P_{n-1}(x) - P_n(x)P_{n-1}(x) \\ &= P_n^2(x) - P_{n+1}(x)P_{n-1}(x). \end{aligned} \tag{A4}$$

Appendix B

Here we list the first few Catalan polynomials with closed form given by (3); they are factored where possible:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= 1, \\ P_2(x) &= 1 - x, \\ P_3(x) &= 1 - 2x, \\ P_4(x) &= 1 - 3x + x^2, \\ P_5(x) &= 1 - 4x + 3x^2 = (1-x)(1-3x), \\ P_6(x) &= 1 - 5x + 6x^2 - x^3, \\ P_7(x) &= 1 - 6x + 10x^2 - 4x^3 = (1-2x)(1-4x+2x^2), \\ P_8(x) &= 1 - 7x + 15x^2 - 10x^3 + x^4 = (1-x)(1-6x+9x^2-x^3), \\ P_9(x) &= 1 - 8x + 21x^2 - 20x^3 + 5x^4 = (1-3x+x^2)(1-5x+5x^2), \\ P_{10}(x) &= 1 - 9x + 28x^2 - 35x^3 + 15x^4 - x^5, \\ P_{11}(x) &= 1 - 10x + 36x^2 - 56x^3 + 35x^4 - 6x^5 \\ &= (1-x)(1-2x)(1-3x)(1-4x+x^2), \end{aligned}$$

$$
\begin{aligned}
P_{12}(x) &= 1-11x+45x^2-84x^3+70x^4-21x^5+x^6, \\
P_{13}(x) &= 1-12x+55x^2-120x^3+126x^4-56x^5+7x^6 \\
&= (1-5x+6x^2-x^3)(1-7x+14x^2-7x^3), \qquad \text{(B1)}
\end{aligned}
$$

etc.

Appendix C

We give here a couple of other examples of the phenomenon of masked periodicity.

Example E: $p=-\sqrt{3}i$, $q=-1$ ($p\in\mathbf{C}$, $q\in\mathbf{R}$)
Noting that

$$
\mathbf{A}^3(-\sqrt{3}i,-1) \;=\; \begin{pmatrix} \sqrt{3}i & -2 \\ -2 & -\sqrt{3}i \end{pmatrix} \;\neq\; \mathbf{I}_2, \qquad \text{(C1)}
$$

then with an initial values vector $\mathbf{w}=(-\frac{1}{2}(1+\sqrt{3}i)a,a)^T$ for which $\mathbf{A}^3\mathbf{w}=\mathbf{w}$, the sequence $\{w_n(a,-\frac{1}{2}(1+\sqrt{3}i)a;-\sqrt{3}i,-1)\}_0^\infty=\{a,-\frac{1}{2}(1+\sqrt{3}i)a, -\frac{1}{2}(1-\sqrt{3}i)a,\ldots\}$ has period 3, and $F(-\sqrt{3}i,-1,3)=0$. This starting values specific sequence, however, is masked by the (lower frequency) period 6 sequence $\{w_n(a,b;-\sqrt{3}i,-1)\}_0^\infty=\{a,b,a-\sqrt{3}bi,-2b-\sqrt{3}ai, -2a+\sqrt{3}bi,b+\sqrt{3}ai,\ldots\}$, with $\mathbf{A}^6(-\sqrt{3}i,-1)=\mathbf{I}_2$ and $[-\sqrt{3}i,-1,6]$ an identity triplet. The Lemma conditions (i),(ii) in this instance demand (noting that $q/p^2=1/3$) $P_5(1/3)=0$ and $P_6(1/3)=-1/27$, and both hold.

Example F: $p=-(1+i)$, $q=i$ ($p,q\in\mathbf{C}$)
We find $\mathbf{A}^4(-(1+i),i)=\mathbf{I}_2$, with associated identity triplet $[-(1+i),i,4]$, which delivers a sequence $\{w_n(a,b;-(1+i),i)\}_0^\infty=\{a,b,-b-(a+b)i,-a+(a+b)i,\ldots\}$ of period 4. On choosing $b=-a$ we observe a (higher frequency) masked period 2 sequence $\{w_n(a,-a;-(1+i),i)\}_0^\infty=\{a,-a,\ldots\}$ with $\mathbf{A}^2\mathbf{w}=\mathbf{w}$ for the initial values vector $\mathbf{w}=(-a,a)^T$ (though $\mathbf{A}^2(-(1+i),i)\neq\mathbf{I}_2$), and $F(-(1+i),i,2)=0$. The Lemma conditions $P_3(i/(1+i)^2)=0$ and $P_4(i/(1+i)^2)=1/(1+i)^4$ have been checked for the said triplet.

We also see that $0=F(-\sqrt{3}i,-1,6)=F(-(1+i),i,4)$, together with (in addition to (42)) the condition

$$
F\left(-\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-1\right)i,\frac{1}{\sqrt{2}}(1+i),8\right) \;=\; 0 \qquad \text{(C2)}
$$

from Example D of Section 3.1.1; this is, of course, to be expected since (given p,q) if $F(p,q,\delta)=0$ for some (minimal) period δ, it follows trivially that $F(p, q, k\delta) = 0$ (integer $k \geq 2$).

Appendix D

In his 1965 paper [3, p.166] Horadam remarked briefly on the geometric type nature of sequences when one of p or q is zero. We can both frame this observation in the context of identity triplets and make some other pertinent remarks.

Case $p = 0$

For $p = 0$ ($q \neq 0$) the recurrence (1) remains second order and generates the sequence $\{w_n(a, b; 0, q)\}_0^\infty = \{a, b, -aq, -bq, aq^2, bq^2, -aq^3, -bq^3, \ldots\}$ comprising two geometric subsequences. If $q = -1$ this becomes the (minimum) period 2 sequence $\{w_n(a, b; 0, -1)\}_0^\infty = \{a, b, \ldots\}$, with $[0, -1, 2]$ an identity triplet and

$$\mathbf{A}^2(0, -1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \mathbf{I}_2, \tag{D1}$$

as expected. If $q = 1$ then the sequence $\{w_n(a, b; 0, 1)\}_0^\infty = \{a, b, -a, -b, \ldots\}$ of (minimum) period 4 results, for which

$$\mathbf{A}^4(0, 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = \mathbf{I}_2, \tag{D2}$$

and $[0, 1, 4]$ is an identity triplet.

Case $q = 0$

For $q = 0$ ($p \neq 0$), Horadam had in mind the geometric sequence $\{a, ap, ap^2, ap^3, ap^4, \ldots\}$ as the consequence of a contraction of (1) to the first order relation $w_n = pw_{n-1}$ ($n \geq 1$) with a single initial value $w_0 = a$. It is for this reason of reduction in order that, whilst if $p = 1$ the sequence $\{w_n(a, -; 1, 0)\}_0^\infty = \{a, a, \ldots\}$ delivered is of (minimum) period 1, we see

$$\mathbf{A}^1(1, 0) = \mathbf{A}(1, 0) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq \mathbf{I}_2 \tag{D3}$$

and $[1, 0, 1]$ is not an identity triplet (in fact the matrix $\mathbf{A}(1, 0)$ is idempotent, so when raised to any integer power the resulting matrix will never take the form of the identity matrix). In a similar fashion, the (minimum) period 2 sequence $\{w_n(a, -; -1, 0)\}_0^\infty = \{a, -a, \ldots\}$ arising from $p = -1$ produces no identity triplet $[-1, 0, 2]$ either, since $\mathbf{A}^2(-1, 0) \neq \mathbf{I}_2$.

In Section 1 we noted that the cyclicity of the period 3 and 6 sequences

$\{w_n(a,b;\pm 1,1)\}_0^\infty$ (4) was mentioned in Horadam's paper [3]. It is interesting to see that these were discussed almost two decades later in a geometric context by Bergum [23, p.24] who explained why points in the x, y plane of the form $(w_n(a,b;\pm 1,1), w_{n+1}(a,b;\pm 1,1))$ (with co-ordinates of successive sequence terms) lay on respective ellipses $x^2 + y^2 \mp xy = a^2 + b^2 \mp ab$; in addition, such points $(w_n(a,b;0,1), w_{n+1}(a,b;0,1))$ associated with the period 4 sequence referred to above were shown to satisfy the equation of the circle $x^2 + y^2 = a^2 + b^2$. The number of distinct points lying on each curve matches, of course, the period of the sequence with which it is associated (in the latter case, for instance, there are just four points $(a,b), (b,-a), (-a,-b), (-b,a)$ on the said circle). All loci of points of general type $(w_n(a,b;p,q), w_{n+1}(a,b;p,q))$ are described by conics $qx^2 + y^2 - pxy = (qa^2 + b^2 - pab)q^n$ which specialise to a hyperbola ($p^2 > 4q$), an ellipse or circle ($p^2 < 4q$), or a parabola ($p^2 = 4q$)—this applies to both periodic and non-periodic sequences.

References

[1] Larcombe, P.J. and Fennessey, E.J. (2014). On cyclicity and density of some Catalan polynomial sequences, *Bull. I.C.A.*, **71**, pp.87-93.

[2] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2008). On iterated generating functions for integer sequences, and Catalan polynomials, *Util. Math.*, **77**, pp.3-33.

[3] Horadam, A.F. (1965). Basic properties of a certain generalized sequence of numbers, *Fib. Quart.*, **3**, pp.161-176.

[4] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2012). New closed forms for Householder root finding functions and associated nonlinear polynomial identities, *Util. Math.*, **87**, pp.131-150.

[5] Jarvis, A.F., Larcombe, P.J. and Fennessey, E.J. (2014). Some factorisation and divisibility properties of Catalan polynomials, *Bull. I.C.A.*, **71**, pp.36-56.

[6] Larcombe, P.J., Bagdasar, O.D. and Fennessey, E.J. (2013). Horadam sequences: a survey, *Bull. I.C.A.*, **67**, pp.49-72.

[7] Bagdasar, O.D. and Larcombe, P.J. (2013). On the characterization of periodic complex Horadam sequences, *Fib. Quart.*, **51**, pp.28-37.

[8] Bagdasar, O., Larcombe, P.J. and Anjum, A. (2013). Particular orbits of periodic Horadam sequences, *Oct. Math. Mag.*, **21**, pp.87-98.

[9] Elsayed, E.M. and Iričanin, B.D. (2009). On a max-type and a min-type difference equation, *Appl. Math. Comp.*, **215**, pp.608-614.

[10] Iričanin, B.D. and Elsayed, E.M. (2010). On the max-type difference equation $x_{n+1} = \max\{A/x_n, x_{n-3}\}$, *Disc. Dyn. Nat. Soc.*, Art. I.D. 675413, 13pp.

[11] Alter, R. and Kubota, K.K. (1973). Multiplicities of second order linear recurrences, *Trans. Amer. Math. Soc.*, **178**, pp.271-284.

[12] McGuire, T. (2008/09) On the periodicity of certain recursive sequences, *Fib. Quart.*, **46/47**, pp.350-355.

[13] Puri, Y. and Ward, T. (2001). Arithmetic and growth of periodic orbits, *J. Int. Seq.*, **4**, Article No. 01.2.1, 18pp.

[14] Lau, C.-L. (1977). The periodic generating sequence, *Fib. Quart.*, **15**, pp.178-182.

[15] Melham, R.S. and Shannon, A.G. (1994). Some congruence properties of generalized second-order integer sequences, *Fib. Quart.*, **32**, pp.424-428.

[16] Zhang, Z. (1997). Some identities involving generalized second-order integer sequences, *Fib. Quart.*, **35**, pp.265-268.

[17] Zhang, Z. and Liu, M. (1998). Generalizations of some identities involving generalized second-order integer sequences, *Fib. Quart.*, **36**, pp.327-328.

[18] Hall, M. (1937). Divisors of second-order sequences, *Bull. Amer. Math. Soc.*, **43**, pp.78-80.

[19] Ward, M. (1934). An arithmetical property of recurring series of the second order, *Bull. Amer. Math. Soc.*, **40**, pp.825-828.

[20] Ward, M. (1954). Prime divisors of second order recurring sequences, *Duke Math. J.*, **21**, pp.607-614.

[21] Ballot, C. (1995). Density of prime divisors of linear recurrences, *Mem. Amer. Math. Soc.*, **115** (No. 551), 102pp.

[22] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2012). The Delannoy numbers: three new non-linear identities, *Bull. I.C.A.*, **64**, pp.39-56.

[23] Bergum, G.E. (1984). Addenda to geometry of a generalized Simson's formula, *Fib. Quart.*, **22**, pp.22-28.