On Cyclicity and Density of Some Catalan Polynomial Sequences

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Abstract

We give proofs of cyclic and density properties of some sequences generated by Catalan polynomials, and other observations.

1 Introduction

In this short offering we re-visit—and use as motivation—an observation made empirically in [1, Section 5.1.2, p.18] on a particular periodic sequence generated by the so called Catalan polynomials (further details on which are to be found in [2] and elsewhere), namely, Property (a) of the following Theorem:

<u>**Theorem**</u> *Let* $P_n(x)$ *denote the* $(n+1)$*th Catalan polynomial* $(n \geq 0)$. *The normalised sequence* $\{\sqrt{4x-1}P_n(x)/2x^{(n+1)/2}\}_0^\infty$ *is, for rational* $x > \frac{1}{4}$,

(a) *periodic (of order 6) if* $x = 1$;
(b) *periodic (of order 8) if* $x = 1/2$;
(c) *periodic (of order 12) if* $x = 1/3$;
(d) *otherwise dense in* $[-1, 1]$.

Whilst the sequence of Property (a) is—as noted in [1]—available directly from a cyclotomic polynomial (see Remark 3 later), our proof of the Theorem employs a line of argument which readily yields all properties. Properties (b)-(d) are seen here for the first time.

2 Proof

Our starting point is the closed form for $P_n(x)$ [1, (70), p.17],

$$P_n(x) = \frac{1}{2^{n+1}} \frac{(1+\sqrt{1-4x}\,)^{n+1} - (1-\sqrt{1-4x}\,)^{n+1}}{\sqrt{1-4x}}, \qquad \text{(P1)}$$

which is re-written for convenience as

$$P_n(x) = -\frac{i}{2^{n+1}} \frac{(1+\sqrt{4x-1}\,i\,)^{n+1} - (1-\sqrt{4x-1}\,i\,)^{n+1}}{\sqrt{4x-1}}, \qquad \text{(P2)}$$

valid for $n \geq 0$; x (rational) being greater than $\frac{1}{4}$ ensures $\sqrt{4x-1} > 0$. Consider a line from the origin to the point $z(\theta_x) = 1 + \sqrt{4x-1}\,i$ in (the top right quadrant of) the complex plane, where $\theta_x \in (0, \frac{\pi}{2})$ is the angle between the line and the real axis. With $|z(\theta_x)| = 2\sqrt{x}$ then, noting that

$$1 + \sqrt{4x-1}\,i = 2\sqrt{x}\exp(i\theta_x), \quad 1 - \sqrt{4x-1}\,i = 2\sqrt{x}\exp(-i\theta_x), \qquad \text{(P3)}$$

$P_n(x)$ reduces to

$$P_n(x) = \frac{2x^{(n+1)/2}}{\sqrt{4x-1}} \sin[(n+1)\theta_x], \qquad n \geq 0, \qquad \text{(P4)}$$

after a little simplification; thus,

$$\frac{\sqrt{4x-1}\,P_n(x)}{2x^{(n+1)/2}} = \sin[(n+1)\theta_x], \qquad n \geq 0. \qquad \text{(P5)}$$

Properties (a)-(d) are now delivered easily through consideration of two separate cases imposed on θ_x:

<u>Case I</u>
Let $s \in \mathbf{Q}$ be rational, and suppose $\theta_x = 2\pi s$. Since $\cos(\theta_x) = 1/2\sqrt{x}$ then $\cos(2\theta_x) = 2\cos^2(\theta_x) - 1 = 2(1/4x) - 1 = 1/2x - 1$ which, as x is rational, itself takes rational values. At this point we recall the below Lemma (stated and established in an accompanying article [3] as Lemma 1):

<u>Lemma</u> *Let $q, t \in \mathbf{Q}$. If $t = \cos(q\pi)$ then $t \in \{0, \pm\frac{1}{2}, \pm1\}$.*

We know $\cos(2\theta_x) \in \mathbf{Q}$, and also $2\theta_x = q\pi$ (where $q = 4s \in \mathbf{Q}$), upon which we can apply the Lemma and write

$$\cos(2\theta_x) \;=\; \frac{1}{2x} - 1 \;\in\; \left\{0, \pm\frac{1}{2}, \pm 1\right\}. \qquad \text{(P6)}$$

It is trivially seen that only the values $0, -\frac{1}{2}, \frac{1}{2}$ of the given set correspond to valid values of x which are, respectively, $x = \frac{1}{2}, 1, \frac{1}{3}$; these lead to Properties (a)-(c).

In the first instance, therefore, consider the value $x = 1$, whence $\theta_x = \cos^{-1}(1/2\sqrt{x}) = \cos^{-1}(1/2) = \pi/3$ and (P5) reads

$$\frac{\sqrt{3}\,P_n(1)}{2} = \sin[(n+1)\pi/3], \qquad n \geq 0, \qquad \text{(P7)}$$

i.e.,

$$\begin{aligned}\left\{\frac{\sqrt{3}\,P_n(1)}{2}\right\}_0^\infty &= \left\{\sin\left[(n+1)\frac{\pi}{3}\right]\right\}_0^\infty \\ &= \left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \ldots\right\}, \qquad \text{(P8)}\end{aligned}$$

which is the period 6 sequence of Property (a) of the Theorem; in other words (see O.E.I.S. Sequence No. A010892),

$$\{P_n(1)\}_0^\infty \;=\; \{1, 1, 0, -1, -1, 0, \ldots\}. \qquad \text{(P9)}$$

With regard to Properties (b),(c), we leave it as a straightforward exercise to show that the remaining values $x = \frac{1}{2}$ and $x = \frac{1}{3}$ produce, by the same process, cyclic (resp., period 8 and 12) sequences which simplify to

$$\{(\sqrt{2})^n P_n(1/2)\}_0^\infty \;=\; \{1, \sqrt{2}, 1, 0, -1, -\sqrt{2}, -1, 0, \ldots\} \qquad \text{(P10)}$$

and

$$\begin{aligned}&\{(\sqrt{3})^n P_n(1/3)\}_0^\infty \\ &\quad = \{1, \sqrt{3}, 2, \sqrt{3}, 1, 0, -1, -\sqrt{3}, -2, -\sqrt{3}, -1, 0, \ldots\}. \qquad \text{(P11)}\end{aligned}$$

Case II

Let r be irrational, and suppose $\theta_x = 2\pi r$. We define a sequence $\mathcal{T} = \{t_n\}_0^\infty$ according to the linear recurrence

$$\begin{aligned} t_0 &= r, \\ t_n &= t_{n-1} + r \pmod 1), \qquad n \geq 1, \qquad \text{(P12)}\end{aligned}$$

and let the set $T = \{t_i : i \in \mathbf{N}\}$ contain all elements of it. It is known that T is dense in the interval $[0,1]$ (*i.e.*, for $\alpha \in [0,1]$ there exists a subsequence $\mathcal{T}_s$ of $\mathcal{T}$, with general term t_{i_j}, say, such that $t_{i_j} \to \alpha$ as $j \to \infty$; w.r.t. notation, i_j values here are drawn from a subset of $\mathbf{N}$). Consider now the set $T^* = \{\sin(2\pi t_i) : i \in \mathbf{N}\}$. It is a simple matter to show (by 'telescoping') that, mod 1, $t_n = (n+1)r$ (or equivalently that, mod 2π, $2\pi t_n = 2\pi(n+1)r$) for $n \in \mathbf{N}$, whence $\sin(2\pi t_n) = \sin[2\pi(n+1)r]$ so that

$$T^* = \{\sin[2\pi(n+1)r] : n \in \mathbf{N}\}. \qquad \text{(P13)}$$

We argue that T^* is dense in $[-1,1]$. For real $\beta \in [-1,1]$ choose $\alpha_\beta \in [0,1]$ such that $\sin(2\pi\alpha_\beta) = \beta$, and let t_{i_j} be such that $t_{i_j} \to \alpha_\beta$ as $j \to \infty$. By the continuity of the sine function it follows that $\sin(2\pi t_{i_j}) \to \beta$ as $j \to \infty$. Now, all terms in the sequence $\{\sin(2\pi t_{i_j})\}_{i_j}$ are in the set T^*, and so T^* forms a sequence

$$\mathcal{T}^* = \{\sin[2\pi(n+1)r]\}_{n \in \mathbf{N}} = \{\sin[(n+1)\theta_x]\}_0^\infty \qquad \text{(P14)}$$

which is dense over $[-1,1]$; this proves Property (d).□

We complete the paper with some final remarks (see also the Appendix).

Remark 1 The periodic zeros which occur in each of $P_n(1), P_n(1/2)$ and $P_n(1/3)$ are evident, and we have re-discovered the results

$$0 = P_{3n+2}(1) = P_{4n+3}\left(\frac{1}{2}\right) = P_{6n+5}\left(\frac{1}{3}\right), \qquad n \geq 0, \qquad (1)$$

established in [3] as equations (8) by another means.

Remark 2 We make some observations concerning recurrence formulas governing the cyclic sequences seen in this paper. Firstly, as noted by Wolfram in his tome [4, Sequence (c), p.128], the elements of the period 6 sequence (P9) can (given $P_0(1) = P_1(1) = 1$) be obtained recursively by the relationship $P_n(1) = P_{n-1}(1) - P_{n-2}(1)$ $(n \geq 2)$. This is also immediate from the more general recurrence [1, (69), p.17] for the Catalan polnomials

$$0 = xP_{n-2}(x) - P_{n-1}(x) + P_n(x); \qquad P_0(x) = P_1(x) = 1, \qquad (2)$$

based on which it is an easy matter to show that the period 8 and 12 sequences (P10),(P11) are generated by the following recurrence schemes:

$$\begin{aligned} a_n &= \sqrt{2}a_{n-1} - a_{n-2}; & a_0 = 1, \ a_1 = \sqrt{2}, \\ a_n &= \sqrt{3}a_{n-1} - a_{n-2}; & a_0 = 1, \ a_1 = \sqrt{3}. \end{aligned} \qquad (3)$$

We have found that, for integer $s \geq 1$, the recursion $a_n = \sqrt{s}\,a_{n-1} - a_{n-2}$ (given $a_0 = 1$, $a_1 = \sqrt{s}$) produces a sequence $\{a_n\}_0^\infty = \{(\sqrt{s})^n P_n(\frac{1}{s})\}_0^\infty$; computational tests confirm that, as expected, it is *only* the particular values $s = 1, 2, 3$ which generate *periodic* sequences (with period $s(s-1)+6$, or $2\pi/\cos^{-1}(\sqrt{s}/2)$).[1] Sequences of this kind fall, of course, within the broad class of those governed by a general recurrence $a_n = pa_{n-1} - qa_{n-2}$ seemingly first studied seriously in the 1960s and on which a surprisingly considerable literature exists.

Remark 3 It is noted that the period 6 sequence $\{P_n(1)\}_0^\infty$ is related to the 6th so called cyclotomic polynomial $\Phi_6(x) = x^2 - x + 1$; we see that $1/\Phi_6(x) = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \cdots = \sum_{n\geq 0} P_n(1)x^n$ acts as an ordinary generating function for this sequence (further details on cyclotomic polynomials are available in [3], where they are used to investigate Catalan polynomial factorisation). In addition, we can identify generating functions for the period 8 and 12 sequences, writing $(1 - \sqrt{2}x + x^2)^{-1} = \sum_{n\geq 0} P_n(1/2)(\sqrt{2}x)^n$ and $(1 - \sqrt{3}x + x^2)^{-1} = \sum_{n\geq 0} P_n(1/3)(\sqrt{3}x)^n$.

Remarks 2 and 3 are expanded as appropriate in the Appendix for completeness.

3 Summary

Proofs of cyclic and density properties of some sequences generated by Catalan polynomials have been presented. It is possible that the techniques used here can be applied to other classes of polynomials whose closed forms are, in structure, similar to (P1) for $P_n(x)$—this provides a natural open problem for study.

Appendix

In this Appendix we add some contextual details to Section 2 Remarks 2,3.

As stated in Remark 2 linear second order recurrences, and the sequences derived therefrom, began to interest researchers greatly in the 1960s where—building on isolated *ad hoc* work dating from the late 19th and early 20th centuries—a more systematic study was started by A.F. Horadam, and

[1] Whilst not periodic, the sequence from the case $s = 4$ is worth mentioning as it is easily explained: we have $\{a_n\}_0^\infty = \{2^n P_n(\frac{1}{4})\}_0^\infty = \{2^n \cdot (n+1)/2^n\}_0^\infty = \{n+1\}_0^\infty = \{1, 2, 3, 4, \ldots\}$, the sequence of positive natural numbers (the closed form for $P_n(\frac{1}{4})$ is taken from [1, Remark 2, p.18]).

one or two of his contemporaries, which has continued since. In 1965 Horadam [5, (6), p.437] considered initially a recurrence $u_n = pu_{n-1} - qu_{n-2}$ which, with initial values $u_0 = 1$, $u_1 = p$ delivers a sequence $\{u_n\}_0^\infty = \{u_n(p,q)\}_0^\infty$ containing our three period 6,8 and 12 sequences as the instances $\{u_n(1,1)\}_0^\infty, \{u_n(\sqrt{2},1)\}_0^\infty$ and $\{u_n(\sqrt{3},1)\}_0^\infty$. Additionally, he listed the ordinary generating function in the general case as (see (10), p.438)

$$\sum_{n-0}^{\infty} u_n x^n = \frac{1}{1-px+qx^2}, \tag{A1}$$

which recovers the cyclic sequence generators given in Remark 3. In [5], and also in another paper from the same year [6, (1.5), p.161], Horadam examined the scenario for which *arbitrary* initial values $w_0 = a$, $w_1 = b$ give rise to a more general sequence—written $\{w_n\}_0^\infty = \{w_n(a,b;p,q)\}_0^\infty$ and with terms $\{a, b, pb - qa, (p^2 - q)b - pqa, \ldots\}$—from the same recurrence $w_n = pw_{n-1} - qw_{n-2}$ (the most famous specialisations being the Fibonacci sequence $\{w_n(1,1;1,-1)\}_0^\infty$ and the Lucas sequence $\{w_n(2,1;1,-1)\}_0^\infty$); this sequence is known as a *Horadam sequence*, on which a first literature survey was published last year [7]. Interestingly, Horadam remarked (albeit briefly) [6, (2.35),(2.36), p.166] on the self-repeating nature of the period 3 sequence $\{w_n(a,b;-1,1)\}_0^\infty = \{a, b, -(a+b), \ldots\}$ and the sequence $\{w_n(a,b;1,1)\}_0^\infty = \{a, b, b-a, -a, -b, -(b-a), \ldots\}$ of period 6, the latter describing our sequence $\{P_n(1)\}_0^\infty$ for $a = b = 1$. Other than some very recent work on complex domain orbits (see, for instance, [8-10]), there has—at the time of writing (July 2013)—been an absence of research which specifically examines regular (*i.e.*, non-modulo) cyclicity of the general Horadam sequence $\{w_n(a,b;p,q)\}_0^\infty$ or instances thereof. We note in passing that an open problem, too, is the extension of theory to analyse the possible periodic behaviour of (real or complex) sequences generated by a linear Horadam type recurrence of order greater than two.

Horadam [6] looked at both linear and non-linear properties of the sequence $\{w_n(a,b;p,q)\}_0^\infty$, whilst in the forerunner article [5] his analysis covered kth powers $(w_n)^k$ of sequence terms w_n; Carlitz—motivated by parallel work by Riordan—had similarly looked at the sequences $\{(u_n)^k(p,q)\}_0^\infty$ a little while before [11]. With reference to our three periodic sequences (P9)-(P11), even powers of sequence elements produce further integer sequences, the first of which are

$$\begin{aligned} \{P_n^2(1)\}_0^\infty &= \{1,1,0,1,1,0,\ldots\}, \\ \{2^n P_n^2(1/2)\}_0^\infty &= \{1,2,1,0,1,2,1,0,\ldots\}, \\ \{3^n P_n^2(1/3)\}_0^\infty &= \{1,3,4,3,1,0,1,3,4,3,1,0,\ldots\}, \end{aligned} \tag{A2}$$

with period 3,4,6,[2] and their generating functions are, in order, $(1+x)/(1-x^3)$, $(1+x)/(1-x+x^2-x^3)$ and $(1+x)/(1-2x+2x^2-x^3)$; we do not pursue this notion any further here.

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[2] Sequence No. A011655 is $\{P_n^2(1)\}_2^\infty$, Sequence No. A007877 is $\{2^n P_n^2(1/2)\}_3^\infty$ and Sequence No. A131027 is $\{3^n P_n^2(1/3)\}_2^\infty$ on the O.E.I.S.

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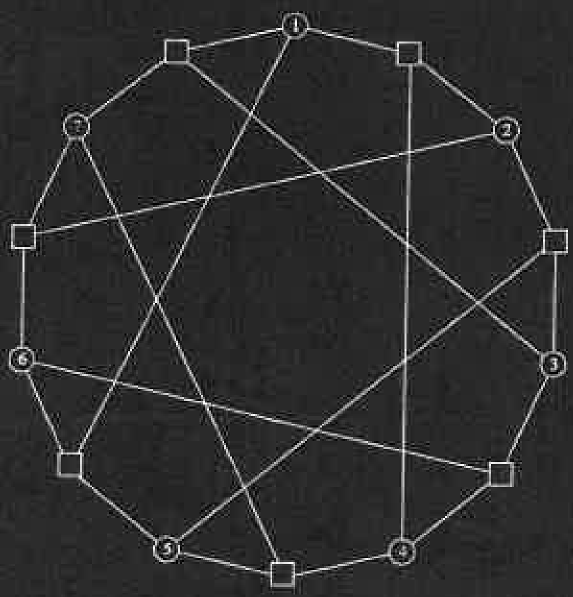

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Commemorating 200 Years Since the Birth of Eugène Charles Catalan

Guest Editor
Peter J. Larcombe

Dedication

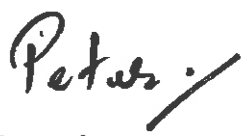
This Special May 2014 Bulletin Issue is Dedicated to the Memory of
David R. French ('Frenchy')
1943—2014

The Catalan sequence has an almost unparalleled ubiquity in discrete mathematics, arising as, or in, the solution of a wide variety of apparently disparate and unconnected counting problems. Throughout the major part of the 19th century the accepted version of its discovery linked the initial identification of the sequence to Leonhard Euler, who in 1751 wrote of its elements as providing solutions to the so called ***triangulated decompositions of polygons***—a problem which is today well known and through which the Catalan sequence was to eventually bear the name of Catalan himself, seemingly after a flurry of activity (by Catalan and some contemporaries) during the 1830s and 1840s. This false attribution (and others) continued until 1988 when a Chinese historian, J. Luo, detailed a new context as evidence of an even earlier awareness of the Catalan sequence by the scholar Antu Ming (who during the first half of the 1700s examined, via geometrical considerations, a certain type of infinite series containing Catalan numbers).

From such beginnings well over 250 years ago, the Catalan sequence has continued to make regular appearances in the literature—sometimes in surprising ways—whilst the Catalan numbers have interesting mathematical properties in their own right which link with other integer sequences. My own personal interest in the Catalan sequence took off when it arose in an enumeration problem on which I was working with an undergraduate final year student in the mid 1990s (strangely, it took many years for this work to be disseminated), and—after the assimilation and translation of the relevant material—I wrote, and co-wrote, a series of short pieces on the origins of the Catalan sequence in an attempt to clarify that part of its history. Since then both Catalan and the Catalan numbers have at times figured in my work, most recently through the so called Catalan polynomials which I discovered with a Ph.D. student (James Clapperton) and great friend Dr. Eric Fennessey (in our study of iterated generating functions) and which form the basis of my joint contributions to this Special Issue. I am, of course, not alone in my Catalan-related pursuits. Professor Richard P. Stanley, for instance, has aptly termed an extreme enthusiasm for all matters Catalan as "Catalania" ("Catalan mania"), a 'condition' whose 'sufferers' will undoubtedly recognise! Richard himself keeps a wonderful Catalan Addendum to Volume 2 of his well known book *Enumerative Combinatorics* active as an up-to-date resource for researchers in which he details new interpretations and problems, and Professor Thomas Koshy has been moved to write a stand alone undergraduate text *Catalan Numbers with Applications* for a less specialised readership (see overleaf for more details on these books). Each, in its own particular way, serves the mathematical community well, along with the numerous articles which have, over the years, formed a substantial body of work on the Catalan sequence and secured its place at the forefront of the world of integer sequences.

One wonders what Catalan—who as well as being politically active was quite eclectic in his mathematical endeavours—would have made of the way the sequence has captivated academics eager to understand its fundamental nature and application; certainly, it is testimony to the importance of the Catalan numbers that so many people, at all academic levels, continue to develop and often retain an interest in them, and there is no sign of this ending. It is, therefore, a great pleasure to write this Foreword in my capacity as Guest Editor, as the I.C.A. formally celebrates both the significant and longstanding impact of the Catalan sequence within discrete mathematics. The invited contributions on offer here are as varied as they are interesting, forming a timely and fitting tribute to Catalan and the Catalan sequence.

Enjoy !



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Major Contributions to the Literature on Catalan Numbers by Stanley and Koshy

R.P. Stanley (1999). "Enumerative Combinatorics", Volume 2 (Cambridge Studies in Advanced Mathematics No. 62), Cambridge University Press, Cambridge, U.K.
Some useful background information on the Catalan numbers (with references) appears in the *Notes* section at the end of Chapter 6 of this advanced level book, with the subsequent Exercises 6.19 offering a number of combinatorial illustrations. Stanley continues to update the original presentation in the textbook with an "EC2 Supplement" (available from his M.I.T. homepage) which contains errata, updates a nd new material. In addition, a "Catalan Addendum" offers new problems related to Catalan numbers, with solutions, reflecting his deep and enduring interest in them and a determination to see them disseminated; Catalan interpretations in the Addendum currently stand at over 200 in number, the collation of which is a commendable achievement on the part of Stanley.

T. Koshy (2009). "Catalan Numbers with Applications", Oxford University Press, New York, U.S.A.
Koshy's text is aimed at a broad readership (of mathematical amateurs, high school students/teachers, and both undergraduate and postgraduate level students), in which he pulls together and catalogues many different aspects of the Catalan sequence and its numerous contexts. The book—as the author rightly states—is the first to collect and present an orderly treatise on the various occurrences, applications and properties of the Catalan numbers, and Koshy draws on a multitude of reference material to create a very useful resource.

Some Other Works of Note on Catalan

In 1996 the Société Belge des Professeurs de Mathématique d'Expression Française (Mons, Belgium) published "Eugène Catalan: Géomètre sans Patrie, Républicain sans République", a 200+ page book by F. Jongmans on the life and work of Catalan. [Prior to this, and as a precursor, the author had contributed a chapter (Chapter 3, pp.23-41) with the same title in a publication "Regards Sur 175 Ans de Science à l'Université de Liège 1817-1992" (Ed. A.-C. Bernès) which was produced in 1992 under the auspices of the University's Centre d'Histoire des Sciences et des Techniques to mark this period of general scientific activity at the university.]

Other works of note are the articles "Eugène Catalan and the Rise of Russian Science" (*Acad. Roy. Belg. Bull. Class. Sci.*, **2** (1991), pp.59-90) by P.L. Butzer and F. Jongmans, "Les Relations Épistolaires Entre Eugène Catalan et Ernesto Cesàro" (*ibid.*, **10** (1999), pp.223-271) by Butzer *et al.*, and "Quelques Pièces Choisies dans la Correspondance d'Eugène Catalan" (*Bull. Soc. Roy. Sci. Liège*, **50** (1981), pp.287-309) by Jongmans. All bar the final reference are predated by about a century by P. Mansion's "Notice sur les Travaux Mathématiques de Eugène-Charles Catalan" which appeared in *Ann. l'Acad. Roy. Sci. Lett. Beaux-Arts Belg.* in 1896 (**62**, pp.115-172).